- Integration:

- We shall discuss the problem of finding a function $y = F(x)$, whose derivative is given by the equation:

$$\frac{dy}{dx} = f(x) \qquad \cdots (1)$$

where eq.(1) usually called a differential equation.

- **Defn.:** A function $y = F(x)$ is called a solution of the differential equation (1) if:

$$\frac{d}{dx} F(x) = f(x)$$

also, $F(x)$ is called an antiderivative of $f(x)$.

To solve eq.(1) it means to find all the functions that are antiderivatives of $f(x)$.

- **Theorem (1):** If $F(x)$ is an antiderivative of $f(x)$, then $F(x)+c$ is also an antiderivative, where c is any constant.

Proof:

Since $F(x)$ is an antiderivative of $f(x)$, then:

$$\frac{d}{dx} F(x) = f(x)$$

Now,

$$\frac{d}{dx}[F(x)+c] = \frac{d}{dx} F(x) + \frac{d}{dx} c = f(x) + 0 = f(x)$$

$\therefore F(x)+c$ is an antiderivative.

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Note: From theorem (1), we may say that, if $F(x)$ is any solution of the equation $\frac{dy}{dx} = f(x)$, then all solutions are given by the formula:

$$y = F(x) + c \quad \cdots (2)$$

- Indefinite Integral:

The set of all antiderivatives «or solutions» of $f(x)$ is called the indefinite integral of f w.r.t x and denoted by:

$$y = \int f(x)\,dx = F(x) + c \quad \cdots (3)$$

where the symbol $\int$ is called an integral sign; the function f is called the integrand of the integral, c is called the Constant of integration, and dx tell us that the variable of integration is x.

Ex. (1): Solve the differential equation $\frac{dy}{dx} = 3x^2$

$\star$

since $\frac{d}{dx}x^3 = 3x^2 \Rightarrow y = \int 3x^2\,dx = x^3 + c$

Ex. (2): Solve: $\frac{dy}{dx} = \sin x \cdot \cos x$

$\star$

since $\frac{d}{dx}\left(\frac{1}{2}\sin^2 x\right) = \sin x \cdot \cos x \Rightarrow$

$$y = \int \sin x \cdot \cos x \cdot dx = \frac{1}{2}\sin^2 x + c$$

- Definite Integrals:

- The integral $\int_a^b f(x)\,dx$ is called the definite integral of $f(x)$ over the interval $[a,b]$.

- The Fundamental Theorem of Integral Calculus:

- If $f(x)$ is continuous function on $[a,b]$ and $F(x)$ is any antiderivative of $f(x)$ over $[a,b]$, then:

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

- Properties of Definite Integrals:

If $f(x)$ is continuous function on $[a,b]$, then:

1- $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

2- $\int_a^a f(x)\,dx = 0$

3- $\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx$, where k is constant.

4 $\int_a^b \{f_1(x) + f_2(x) + \cdots + f_n(x)\}\,dx = \int_a^b f_1(x)\,dx + \int_a^b f_2(x)\,dx + \cdots + \int_a^b f_n(x)\,dx$

5- $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, for any $c \in [a,b]$

- Some Integration Formulas:

If $u = u(x)$, then:

1- $\int \frac{du}{dx}\, dx = u + c.$

2- $\int a.u.dx = a \int u.dx$, where a is constant.

3- $\int \{u_1 + u_2 + \cdots + u_n\} dx = \int u_1 dx + \int u_2 dx + \cdots + \int u_n dx$

4- $\int u^n . \frac{du}{dx} . dx = \frac{u^{n+1}}{n+1} + c \quad , n \neq -1$

Ex.(3): Evaluate: $I = \int (3x^4 - 2x^3 + x^{1/2} + 2x^{-2} + \sqrt{2})\, dx$

$$I = 3\int x^4 dx - 2\int x^3 dx + \int x^{1/2} dx + 2\int x^{-2} dx + \sqrt{2} \int dx$$

$$= \frac{3}{5}x^5 - \frac{1}{2}x^4 + \frac{2}{3}x^{3/2} - 2x^{-1} + \sqrt{2}x + c$$

Ex.(4): Evaluate: $I = \int \sqrt{(3x-1)^3}\, dx$

$$I = \int (3x-1)^{3/2} dx = \frac{1}{3}\int (3x-1)^{3/2} \cdot 3\, dx = \frac{1}{3} \cdot \frac{2}{5} (3x-1)^{5/2} + c$$

$$= \frac{2}{15}(3x-1)^{5/2} + c$$

Ex.(5): Evaluate $I = \int (x^2+1)^3 . 2x\, dx$

$$I = \frac{1}{4}(x^2+1)^4 + c$$

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Ex.(6): Evaluate $I=\int_{-3}^{2}(6-x-x^2)\,dx$

$$I=\left(6x-\frac{1}{2}x^2-\frac{1}{3}x^3\right)\Big|_{-3}^{2}=\left(12-2-\frac{8}{3}\right)-\left(-18-\frac{9}{2}+9\right)=\frac{125}{6}$$

Ex.(7): Find $I=\int_0^{\pi}\sin x\,dx=-\cos x\Big|_0^{\pi}=-(\cos\pi-\cos 0)=2$

Ex.(8): Find $I=\int_0^{\pi/6}\frac{\sin 2x}{\cos^2 2x}\,dx$

$$I=\frac{-1}{2}\int_0^{\pi/6}\cos^{-2}2x\cdot(-2)\cdot\sin 2x\,dx=\frac{-1}{2}\frac{(\cos 2x)^{-1}}{-1}\Big|_0^{\frac{\pi}{6}}$$

$$=\frac{1}{2}\cdot\frac{1}{\cos 2x}=\frac{1}{2}\sec 2x\Big|_0^{\pi/6}$$

$$=\frac{1}{2}\left[\sec\frac{\pi}{3}-\sec 0\right]=\frac{1}{2}(2-1)=\frac{1}{2}$$

Ex.(9): If $f(x)$ is Continuous function, show that:

$$\int_0^1 f(x)\,dx=\int_0^1 f(1-t)\,dt$$

Let $x=1-t \Rightarrow dx=-dt$

at $x=0 \Rightarrow t=1$

at $x=1 \Rightarrow t=0$

then:

$$\int_0^1 f(x)\,dx=\int_1^0 f(1-t)(-dt)=-\int_1^0 f(1-t)\,dt=\int_0^1 f(1-t)\,dt$$

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- Method [1]: "Integration by Substitution"

- The goal of this method is to transform the integral into a standard form.

To evaluate the integral $I=\int f[g(x)]\cdot g'(x)\cdot dx$, carry out the following steps:

1- Substitute $u=g(x)$, then $du=g'(x)\,dx$ to obtain $I=\int f(u)\,du$.
2- Evaluate $I=\int f(u)\,du$ by integrating w.r.t u.
3- Replace u by $g(x)$ in the final result.

Ex.(1): Evaluate $I=\int \frac{dx}{\sqrt[3]{1-2x}}$

$I=\int (1-2x)^{-\frac{1}{3}}\,dx$, let $u=1-2x \Rightarrow du=-2dx$

$$I=\frac{-1}{2}\int -2(1-2x)^{-\frac{1}{3}}dx=\frac{-1}{2}\int(-2)(1-2x)^{-\frac{1}{3}}dx=\frac{-1}{2}\int u^{-\frac{1}{3}}du$$

$$=\frac{-1}{2}\frac{u^{2/3}}{2/3}+c=\frac{-3}{4}(1-2x)^{2/3}+c$$

Ex.(2): Find $I=\int \sin^2 5x\cos 5x\,dx$

let $u=\sin 5x \Rightarrow du=5\cos 5x\,dx$

$$\therefore I=\frac{1}{5}\int 5\sin^2 5x\cdot\cos 5x\cdot dx=\frac{1}{5}\int u^2\,du=\frac{1}{5}\frac{u^3}{3}+c$$

$$=\frac{1}{15}\sin^3 5x+c$$

Ex.(3): Evaluate the following integrals:

1- $I=\int x\, e^{x^2+1}\, dx$, Let $u=x^2+1 \Rightarrow du=2x\,dx$

$$I=\frac{1}{2}\int e^u\, du=\frac{1}{2}e^u+c=\frac{1}{2}e^{x^2+1}+c$$

2- $I=\int \frac{\cosh 3x}{4+\sinh 3x}\, dx$, Let $u=4+\sinh 3x \Rightarrow du=3\cosh 3x$

$$I=\frac{1}{3}\int \frac{du}{u}=\frac{1}{3}\ln u+c=\frac{1}{3}\ln\left[4+\sinh 3x\right]+c$$

3- $I=\int \frac{\cosh 3x}{4+\sinh^2 3x}\, dx$, Let $u=\sinh 3x \Rightarrow du=3\cosh 3x$

$$I=\frac{1}{3}\int \frac{du}{4+u^2}=\frac{1}{3}\cdot\frac{1}{2}\tan^{-1}\frac{u}{2}+c=\frac{1}{6}\tan^{-1}\left[\frac{\sinh 3x}{2}\right]+c$$

4- $I=\int \frac{dx}{\sqrt{4-9x^2}}=\int \frac{dx}{\sqrt{4-(3x)^2}}$, Let $u=3x \Rightarrow du=3$

$$I=\frac{1}{3}\int \frac{du}{\sqrt{2^2-u^2}}=\frac{1}{3}\sin^{-1}\frac{u}{2}+c=\frac{1}{3}\sin^{-1}\frac{3x}{2}+c$$

5- $I=\int \tan^3 3x\cdot \sec^2 3x\cdot dx$, Let $u=\tan 3x \Rightarrow du=3\sec^2 3x$

$$I=\frac{1}{3}\int u^3\, du=\frac{1}{3}\cdot\frac{u^4}{4}+c=\frac{1}{12}\tan^4 3x+c$$

6- $I=\int \frac{\sin^2 2x}{1+\cos 2x}\, dx=\int \frac{1-\cos^2 2x}{1+\cos 2x}\, dx$

$$=\int \frac{(1-\cos 2x)\cdot(1+\cos 2x)}{1+\cos 2x}\, dx=\int (1-\cos 2x)\, dx$$

$$=\frac{1}{2}\left[2x-\sin 2x\right]+c$$

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7- $I = \int \frac{dx}{1+e^x} = \int \frac{dx}{1+\frac{1}{e^{-x}}} = -\int \frac{-e^{-x}dx}{e^{-x}+1} = -\ln(e^{-x}+1)+c$

8- $I = \int \frac{\sqrt{x}}{4+x^3}dx = \int \frac{x^{1/2}}{4+(x^{3/2})^2}dx$, let $u = x^{3/2} \Rightarrow du = \frac{3}{2}x^{1/2}dx$

$I = \frac{2}{3}\int \frac{du}{4+u^2} = \frac{2}{3}\cdot\frac{1}{2}\tan^{-1}\frac{u}{2}+c = \frac{1}{3}\tan^{-1}\frac{x^{3/2}}{2}+c$

9- $I = \int \sec x \cdot dx = \int \sec x \cdot \frac{\sec x+\tan x}{\sec x+\tan x}\cdot dx$

$= \int \frac{\sec^2 x + \sec x\cdot\tan x}{\sec x+\tan x}dx$, let $u = \sec x+\tan x \Rightarrow$

$du = [\sec x\cdot\tan x + \sec^2 x]dx$

$I = \int \frac{du}{u} = \ln u + c = \ln(\sec x+\tan x)+c$

10- $I = \int_0^{1/\sqrt{2}} \frac{x\,dx}{\sqrt{1-x^2}}$, let $u = x^2 \Rightarrow du = 2x\,dx$

at $x = 0 \Rightarrow u = 0$

at $x = \frac{1}{\sqrt{2}} \Rightarrow u = \frac{1}{2}$

$I = \frac{1}{2}\int_0^{1/2} \frac{du}{\sqrt{1-u^2}} = \frac{1}{2}\sin^{-1}u\Big|_0^{1/2} = \frac{1}{2}\left[\sin^{-1}\frac{1}{2} - \sin^{-1}0\right] = \frac{1}{2}\left(\frac{\pi}{6}-0\right) = \frac{\pi}{12}$

11- $I = \int \frac{x\,dx}{x^4+4} = \int \frac{x\,dx}{(x^2)^2+2^2}$, let $u = x^2 \Rightarrow du = 2x\,dx$

$I = \frac{1}{2}\int \frac{du}{u^2+2^2} = \frac{1}{2}\cdot\frac{1}{2}\tan^{-1}\frac{x^2}{2}+c$

12- $I = \int \frac{dx}{x-\sqrt{x}} = \int \frac{dx}{\sqrt{x}(\sqrt{x}-1)}$, let $u = \sqrt{x}-1 \Rightarrow du = \frac{1}{2\sqrt{x}}dx$

$I = 2\int \frac{du}{u} = 2\ln u + c = 2\ln\left[\frac{\sqrt{x}-1}{\sqrt{x}}\right]+c$

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- Method [2]: «Certain Power of Trigonometric and Hyperbolic Integrals»:

- Consider the following integrals forms:

a) $\int \sin^m u \cdot \cos^n u \cdot du$ or $\int \sinh^m u \cdot \cosh^n u \cdot du$

b) $\int \tan^m u \cdot \sec^n u \cdot du$ or $\int \tanh^m u \cdot \operatorname{sech}^n u \cdot du$

c) $\int \cot^m u \cdot \csc^n u \cdot du$ or $\int \coth^m u \cdot \operatorname{csch}^n u \cdot du$

a) for this type, there are three cases:

Case (I): If m is odd and +ive, we factor out $\sin u$ ($\sinh u$) and change the remaining even power of $\sin u$ ($\sinh u$) to $\cos u$ ($\cosh u$) using the identities:

$$\sin^2 u = 1 - \cos^2 u \quad ; \sinh^2 u = \cosh^2 u - 1$$

Ex.(1): Evaluate $I = \int \sin^5 2x \cdot \cos^{-3/2} 2x \cdot dx$

$$I = \int \sin^4 2x \cdot \cos^{-3/2} 2x \cdot \sin 2x \cdot dx$$

$$= \int (1 - \cos^2 2x)^2 \cdot \cos^{-3/2} 2x \cdot \sin 2x \cdot dx$$

$$= \int (1 - 2\cos^2 2x + \cos^4 2x) \cdot \sin 2x \cdot \cos^{-3/2} 2x \cdot dx$$

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$$I=\int[\cos^{-3/2}2x-2\cos^{1/2}2x+\cos^{5/2}2x]\sin 2x\,dx$$

$$=\frac{-1}{2}\left[\frac{\cos^{-1/2}2x}{-1/2}-2\frac{\cos^{3/2}2x}{3/2}+\frac{\cos^{7/2}2x}{7/2}\right]+C$$

$$=\cos^{-1/2}2x+\frac{2}{3}\cos^{3/2}2x-\frac{1}{7}\cos^{7/2}2x+C$$

Case(II): If n is odd and +ive, we factor out $\cos u$ ($\cosh u$) and change the remaining even power of $\cos u$ ($\cosh u$) to $\sin u$ ($\sinh u$) using the identities:

$$\cos^2 u=1-\sin^2 u \;;\; \cosh^2 u=1+\sinh^2 u$$

Ex.(2): Evaluate $I=\int \sinh^4 3x\cdot\cosh^3 3x\cdot dx$

$$I=\int \sinh^4 3x\cdot\cosh^2 3x\cdot\cosh 3x\cdot dx$$

$$=\int \sinh^4 3x\cdot(1+\sinh^2 x)\cdot\cosh 3x\,dx$$

$$=\int[\sinh^4 3x+\sinh^6 3x]\cdot\cosh 3x\,dx$$

$$=\frac{1}{3}\left[\frac{\sinh^5 3x}{5}+\frac{\sinh^7 3x}{7}\right]+C=\frac{1}{15}\sinh^5 3x+\frac{1}{21}\sinh^7 3x+C$$

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Case (III): If both m and n are even and +ive (or one of them zero), we reduce the degree of the expression by using the identities:

$$\sin^2 u = \frac{1-\cos 2u}{2} \quad ; \quad \sinh^2 u = \frac{\cosh 2u - 1}{2}$$

$$\cos^2 u = \frac{1+\cos 2u}{2} \quad ; \quad \cosh^2 u = \frac{\cosh 2u + 1}{2}$$

Ex. (3): Find $I = \int \sin^2 2x \cdot \cos^2 2x \cdot dx$

$$I = \frac{1}{4}\int (1-\cos 4x)\cdot(1+\cos 4x)\cdot dx = \frac{1}{4}\int (1-\cos^2 4x)\, dx$$

$$= \frac{1}{4}\int \left[1 - \frac{1+\cos 8x}{2}\right] dx = \frac{1}{4}\int \left[\frac{1}{2} - \frac{1}{2}\cos 8x\right] dx$$

$$= \frac{1}{4}\left[\frac{x}{2} - \frac{1}{16}\sin 8x\right] + c$$

---

b) In this type, we have two cases:

Case (I): If n is even and +ive, we factor out $\sec^2 u$ ($\operatorname{sech}^2 u$) and change the remaining even power of $\sec u$ ($\operatorname{sech} u$) to $\tan u$ ($\tanh u$) using the identities:

$$\sec^2 u = 1 + \tan^2 u \quad ; \quad \operatorname{sech}^2 u = 1 - \tanh^2 u$$

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Ex.(4): Evaluate $I=\int \operatorname{sech}^4\frac{x}{2}\cdot\tanh^{-1/3}\frac{x}{2}\,dx$

$$I=\int \operatorname{sech}^2\frac{x}{2}\cdot\tanh^{-1/3}\frac{x}{2}\cdot\operatorname{sech}^2\frac{x}{2}\,dx$$

$$=\int\left(1-\tanh^2\frac{x}{2}\right)\tanh^{-1/3}\frac{x}{2}\cdot\operatorname{sech}^2\frac{x}{2}\,dx$$

$$=\int\left(\tanh^{-1/3}\frac{x}{2}-\tanh^{5/3}\frac{x}{2}\right)\cdot\operatorname{sech}^2\frac{x}{2}\,dx$$

$$=2\left[\frac{\tanh^{2/3}(x/2)}{2/3}-\frac{\tanh^{8/3}(x/2)}{8/3}\right]+c=\frac{1}{3}\tanh^{2/3}\frac{x}{2}-\frac{3}{4}\tanh^{8/3}\frac{x}{2}+c$$

Case(II): If m is odd and +ive, we factor out $\sec u\tan u$ $(\operatorname{sech} u\tanh u)$ and change remaining even power of $\tan u(\tanh u)$ to $\sec u(\operatorname{sech} u)$ using the identities:

$$\tan^2 u=\sec^2 u-1\quad;\quad \tanh^2 u=1-\operatorname{sech}^2 u$$

Ex.(5): Evaluate $I=\int \tan^3 2x\cdot\sec^{-1/4}2x\,dx$

$$I=\int \tan^2 2x\cdot\sec^{-5/4}2x\cdot\sec 2x\cdot\tan 2x\,dx$$

$$=\int\left(\sec^2 2x-1\right)\sec^{-5/4}2x\cdot\sec 2x\cdot\tan 2x\,dx$$

$$=\int\left(\sec^{3/4}2x-\sec^{-5/4}2x\right)\cdot\sec 2x\cdot\tan 2x\,dx$$

$$=\frac{1}{2}\left[\frac{\sec^{7/4}2x}{7/4}-\frac{\sec^{-1/4}2x}{-1/4}\right]+c=\frac{2}{7}\sec^{7/4}2x+2\sec^{-1/4}2x+c$$

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c) For this type, there are two cases similar to those of type (b), but we will use the identities:

$$\csc^2 u = \cot^2 u + 1 \quad ; \quad \operatorname{csch}^2 u = \coth^2 u - 1$$

- Method [3]: « Trigonometric Substitutions »:

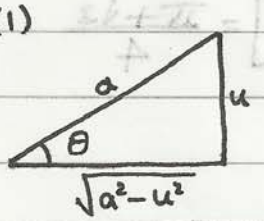
- If the integral involve one of the forms a^2+u^2, $\sqrt{a^2-u^2}$, $\sqrt{a^2+u^2}$, or $\sqrt{u^2-a^2}$. Then the substitutions as follows:

1- If $\sqrt{a^2-u^2}$, let $u = a.\sin\theta \Rightarrow a^2-u^2 = a^2.\cos^2\theta$

2- If $\sqrt{a^2+u^2}$, let $u = a.\tan\theta \Rightarrow a^2+u^2 = a^2.\sec^2\theta$

3- If $\sqrt{u^2-a^2}$, let $u = a.\sec\theta \Rightarrow u^2-a^2 = a^2.\tan^2\theta$

(1)



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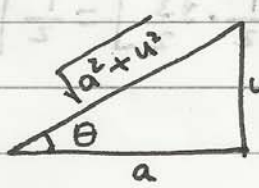

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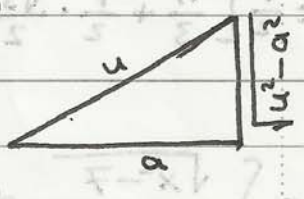

Ex.(1): Evaluate $I = \int \frac{dx}{4+x^2}$

$$I = \frac{1}{2}\tan^{-1}\frac{x}{2} + c$$

or

$x = 2.\tan\theta \Rightarrow \tan\theta = \frac{x}{2} \Rightarrow \theta = \tan^{-1}\frac{x}{2} \Rightarrow$ ~~dx=~~

$dx = 2.\sec^2\theta$

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$$\therefore I = \int \frac{2\sec^2\theta\, d\theta}{4+4\tan^2\theta} = \int \frac{2\sec^2\theta\, d\theta}{4\sec^2\theta} = \frac{1}{2}\int d\theta = \frac{1}{2}\theta + c$$

$$= \frac{1}{2}\tan^{-1}\frac{x}{2} + c$$

Ex.(2): Find I_{π} for the following integrals:

1- $$I = \int_{-\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \sqrt{1-x^2}\, dx$$

$x = \sin\theta \Rightarrow dx = \cos\theta\, d\theta$

at $x = \frac{-1}{2} \Rightarrow \frac{-1}{2} = \sin\theta \Rightarrow \theta = \frac{-\pi}{6}$

at $x = \frac{\sqrt{3}}{2} \Rightarrow \frac{\sqrt{3}}{2} = \sin\theta \Rightarrow \theta = \frac{\pi}{3}$

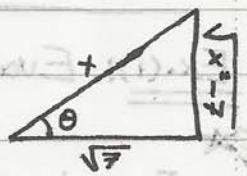
$$I = \int_{\frac{-\pi}{6}}^{\frac{\pi}{3}} \sqrt{1-\sin^2\theta}\cdot\cos\theta\cdot d\theta = \int_{\frac{-\pi}{6}}^{\frac{\pi}{3}} \cos^2\theta\, d\theta = \int_{\frac{-\pi}{6}}^{\frac{\pi}{3}} \frac{1+\cos 2\theta}{2}\, d\theta$$

$$= \frac{1}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_{-\pi/6}^{\pi/3} = \frac{1}{2}\left[\left(\frac{\pi}{3} + \frac{1}{2}\sin\frac{2\pi}{3}\right) - \left(\frac{-\pi}{6} + \frac{1}{2}\sin\left(\frac{-\pi}{3}\right)\right)\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{3} + \frac{1}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\pi}{6} + \frac{1}{2}\cdot\frac{\sqrt{3}}{2}\right] = \frac{1}{2}\left[\frac{\pi}{2} + \frac{\sqrt{3}}{2}\right] = \frac{\pi+\sqrt{3}}{4}$$

2- $$I = \int \frac{\sqrt{x^2-7}}{x}\, dx$$

$x = \sqrt{7}\sec\theta \Rightarrow \sec\theta = \frac{x}{\sqrt{7}} \Rightarrow \theta = \sec^{-1}\frac{x}{\sqrt{7}}$



$dx = \sqrt{7}\sec\theta\cdot\tan\theta\cdot d\theta$

$$I = \int \frac{\sqrt{7\sec^2\theta - 7}}{\sqrt{7}\sec\theta}\cdot \sqrt{7}\sec\theta\cdot\tan\theta\, d\theta = \int \sqrt{7}\tan^2\theta\, d\theta$$

$$= \sqrt{7}\int(\sec^2\theta - 1)\, d\theta = \sqrt{7}\left[\tan\theta - \theta\right] + c$$

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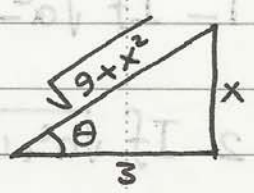
$$\therefore I = \sqrt{7}\left[\tan\left(\sec^{-1}\frac{x}{\sqrt{7}}\right) - \sec^{-1}\frac{x}{\sqrt{7}}\right] + c$$

$$\text{or } I = \sqrt{7}\left[\frac{1}{\sqrt{7}}\sqrt{x^2-7} - \sec^{-1}\frac{x}{\sqrt{7}}\right] + c$$

3- $I = \int x^3\sqrt{9+x^2}\,dx$



$x = 3\tan\theta$

$dx = 3\sec^2\theta.\,d\theta$

$$I = \int 27\tan^3\theta \cdot 3\sec^2\theta \cdot 3\sec\theta\,d\theta$$

$$= \int 243\tan^3\theta.\sec^3\theta.d\theta = 243\int(\sec^2\theta - 1)\sec^2\theta.\sec\theta.\tan\theta\,d\theta$$

$$= 243\int(\sec^4\theta - \sec^2\theta)\sec\theta.\tan\theta.\,d\theta$$

$$= 243\left[\frac{\sec^5\theta}{5} - \frac{\sec^3\theta}{3}\right] + c = 243\left[\frac{1}{5}\left\{\frac{3}{\sqrt{9+x^2}}\right\}^5 - \frac{1}{3}\left\{\frac{3}{\sqrt{9+x^2}}\right\}^3\right] + c$$

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- Method [4]: « Hyperbolic Substitutions »

- As in method [3], the hyperbolic substitutions can be used to the forms a^2+u^2, $\sqrt{a^2-u^2}$, $\sqrt{a^2+u^2}$ or $\sqrt{u^2-a^2}$.

1- If $\sqrt{a^2-u^2}$, Let $u = a.\tanh v$ or $u = a.\operatorname{sech} v$

2- If $\sqrt{a^2+u^2}$, Let $u = a.\sinh v$ or $u = a.\operatorname{csch} v$

3- If $\sqrt{u^2-a^2}$, Let $u = a.\cosh v$ or $u = a.\coth v$

Ex.(1): $I = \int \frac{dx}{x^3\sqrt{x^2+4}}$

Let $x = 2\sinh v \Rightarrow dx = 2\cosh v\, dv$

$$\therefore I = \int \frac{2\cosh v\, dv}{8\sinh^3 v \cdot 2\cosh v} = \int \frac{dv}{8\sinh^3 v} = \frac{1}{8}\int \operatorname{csch}^3 v\, dv$$

it is not easy to find.

We try: $x = 2\operatorname{csch} v \Rightarrow v = \operatorname{csch}^{-1}\frac{x}{2}$

$dx = -2\operatorname{csch} v.\coth v.dv$

$$\therefore I = \int \frac{-2\operatorname{csch} v.\coth v.dv}{8\operatorname{csch}^3 v.2\coth v} = \frac{-1}{8}\int \sinh^2 v.dv$$

$$= \frac{-1}{8}\int\left[\frac{\cosh 2v - 1}{2}\right]dv = \frac{-1}{16}\left[\frac{1}{2}\sinh 2v - v\right]+c$$

$$= \frac{-1}{16}\left[\sinh v.\cosh v - v\right]+c = \frac{-1}{16}\left[\frac{2}{x}.\frac{\sqrt{x^2+4}}{x} - \operatorname{csch}^{-1}\frac{x}{2}\right]+c$$

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Ex(2): Evaluate the following integrals:

1- $I = \int \sec x \cdot dx$ [see method(1)]

let $u = \sec x$

$du = \sec x \cdot \tan x \cdot dx \Rightarrow du = \sec x \cdot \sqrt{\sec^2 x - 1} \cdot dx$

$$I = \int u \cdot \frac{du}{u \cdot \sqrt{u^2-1}} = \int \frac{du}{\sqrt{u^2-1}} = \cosh^{-1} u + c$$

$$= \ln\left[u + \sqrt{u^2-1}\right] + c = \ln\left[\sec x + \sqrt{\sec^2 x - 1}\right] + c$$

$$= \ln\left[\sec x + \tan x\right] + c$$

2- $I = \int \frac{x^3\, dx}{(4-x^2)^{3/2}}$

let $x = 2\tanh v \Rightarrow dx = 2\,\mathrm{sech}^2 v \cdot dv$

$$I = \int \frac{8 \cdot \tanh^3 v \cdot 2\,\mathrm{sech}^2 v \cdot dv}{(4 - 4\tanh^2 v)^{3/2}} = \int \frac{16 \tanh^3 v \cdot \mathrm{sech}^2 v \cdot dv}{(4\,\mathrm{sech}^2 v)^{3/2}}$$

$$= \int \frac{16 \tanh^3 v \cdot \mathrm{sech}^2 v \cdot dv}{8 \cdot \mathrm{sech}^3 v} = 2 \int \tanh^3 v \cdot (\mathrm{sech} v)^{-1} dv$$

$$= 2 \int (1 - \mathrm{sech}^2 v)\, \mathrm{sech}^{-2} v \cdot \mathrm{sech} v \cdot \tanh v \cdot dv$$

$$= 2 \int (\mathrm{sech}^{-2} v - 1)\, \mathrm{sech} v \cdot \tanh v \cdot dv = 2\left[\frac{-(\mathrm{sech} v)^{-1}}{-1} + \mathrm{sech} v\right]$$

$$= 2\left[\cosh v + \mathrm{sech} v\right] + c = 2\left[\frac{2}{\sqrt{4-x^2}} + \frac{\sqrt{4-x^2}}{2}\right] + c$$

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- Method [5]: « Integrals Involving Quadrate Functions »:

- If the integral involve a quadrate function x^2+ax+b, we reduced it to the form u^2+B by completing the squake as follows:

$$x^2+ax+b = x^2+ax+\frac{a^2}{4}+b-\frac{a^2}{4} = \left(x+\frac{a}{2}\right)^2+\left(b-\frac{a^2}{4}\right) = u^2+B$$

where $u = x+\frac{a}{2}$ and $B = b-\frac{a^2}{4}$.

then the solution can be found by Method [3] or [4].

Ex. (1): Evaluate $I = \int \frac{dx}{\sqrt{2x-x^2}}$

$$I = \int \frac{dx}{\sqrt{-(x^2-2x+1-1)}} = \int \frac{dx}{\sqrt{-[(x-1)^2-1]}}$$

$$= \int \frac{dx}{\sqrt{1-(x-1)^2}}$$

let $u = x-1 \Rightarrow du = dx$

$$\therefore I = \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1}u + c = \sin^{-1}(x-1) + c$$

Ex(2): Find $I = \int \frac{(4x+5)\,dx}{(x^2-2x+2)^{3/2}}$

$$I = \int \frac{(4x+5)\,dx}{(x^2-2x+1+1)^{3/2}} = \int \frac{(4x+5)\,dx}{[(x-1)^2+1]^{3/2}}$$

Let $u = x-1 \Rightarrow x = u+1$

$du = dx$

$$\therefore I = \int \frac{[4(u+1)+5]\,du}{(u^2+1)^{3/2}} = \int \frac{4u+9}{(u^2+1)^{3/2}}\,du$$

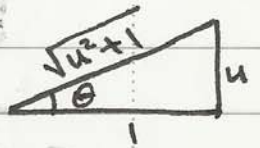
$$I = \int \frac{4u\,du}{(u^2+1)^{3/2}} + \int \frac{9\,du}{(u^2+1)^{3/2}}$$

$$= 2\int 2u\,(u^2+1)^{-3/2}\,du + 9\int \frac{du}{(u^2+1)^{3/2}}$$

$$= 2\,\frac{(u^2+1)^{-1/2}}{-1/2} + 9\int \frac{du}{(u^2+1)^{3/2}}$$

Consider $I_1 = \int \frac{du}{(u^2+1)^{3/2}}$

let $u = \tan\theta \Rightarrow du = \sec^2\theta\,d\theta$



$$I_1 = \int \frac{\sec^2\theta\,d\theta}{[\sec^2\theta]^{3/2}} = \int \cos\theta\,d\theta = \sin\theta + c$$

$$= \frac{u}{\sqrt{u^2+1}} + c$$

$$\therefore I = \frac{-4}{\sqrt{u^2+1}} + \frac{9u}{\sqrt{u^2+1}} + c = \frac{9u-4}{\sqrt{u^2+1}} + c$$

$$= \frac{9(x-1)-4}{\sqrt{(x-1)^2+1}} + c = \frac{9x-13}{(x^2-2x+2)^{1/2}} + c$$

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- Method [6]: « Integration by parts »

- Consider $w = u \cdot v$

$$dw = u \cdot dv + v \cdot du \Rightarrow u \cdot dv = dw - v \cdot du$$

$$\int u \cdot dv = \int dw - \int v \cdot du = w - \int v \cdot du$$

$$\therefore \boxed{\int u \cdot dv = u \cdot v - \int v \cdot du}$$

Ex.(1): Evaluate $I = \int \ln x \, dx$

$$u = \ln x \quad , \quad dv = dx$$
$$du = \frac{1}{x} dx \quad , \quad v = x$$

$$\therefore I = x \cdot \ln x - \int x \cdot \frac{1}{x} dx = x \cdot \ln x - x + c$$

Ex.(2): Evaluate $I = \int \tan^{-1} x \, dx$

$$u = \tan^{-1} x \quad , \quad dv = dx$$
$$du = \frac{dx}{1+x^2} \quad , \quad v = x$$

$$I = x \cdot \tan^{-1} x - \int \frac{x}{1+x^2} dx = x \cdot \tan^{-1} x - \frac{1}{2} \ln(1+x^2) + c$$

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Ex.(3): Find the following integrals:

1- $I = \int x \ln x \, dx$

$u = \ln x$, $dv = x\,dx$

$du = \frac{dx}{x}$, $v = \frac{x^2}{2}$

$$I = \frac{x^2}{2}\ln x - \int \frac{x^2}{2}\cdot\frac{dx}{x} = \frac{x^2}{2}\ln x - \frac{x^2}{4} + c$$

2- $I = \int \sin^3 4x \, dx = \int \sin^2 4x \cdot \sin 4x \cdot dx$

$u = \sin^2 4x$, $dv = \sin 4x\,dx$

$du = 8 \sin 4x \cdot \cos 4x \cdot$, $v = \frac{-1}{4}\cos 4x$

$$I = \frac{-1}{4}\sin^2 4x \cdot \cos 4x - \int \frac{-1}{4}\cos 4x \cdot 8 \sin 4x \cdot \cos 4x \cdot dx$$

$$= \frac{-1}{4}\sin^2 4x \cdot \cos 4x + 2\int \cos^2 4x \cdot \sin 4x \, dx$$

$$= \frac{-1}{4}\sin^2 4x \cdot \cos 4x + \frac{1}{2}\frac{\cos^3 4x}{3} + c$$

3- $I = \int x \cdot e^x \, dx$

$u = x$, $dv = e^x\,dx$

$du = dx$, $v = e^x$

$$I = x \cdot e^x - \int e^x \, dx = x e^x - e^x + c$$

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4- $I = \int x^2 e^x \, dx$

$u = x^2$, $dv = e^x dx$

$du = 2x dx$, $v = e^x$

$$I = x^2 e^x - \int 2x e^x dx = x^2 e^x - 2\left[x e^x - e^x + c\right]$$

- Tabular Integration:

- Consider the integral of the form: $\int f(x) \cdot g(x) \, dx$, in which $f(x)$ can be differentiated repeatedly to become zero and $g(x)$ can be integrated repeatedly without difficulty. Tabular integration saves a great deal of work as natural method candidates for integration by parts.
the method can be illustrated as follows:

$f(x)$ and its derivatives		$g(x)$ and its integrals
$f(x)$	$+$	$g(x)$
$f'(x)$	$-$	$\int g(x) dx = g_1(x)$
$f''(x)$	$+$	$\int g_1(x) dx = g_2(x)$
$f'''(x)$		$\int g_2(x) dx = g_3(x)$
$\vdots$	$\vdots$	$\vdots$
$f^{(n-1)}(x)$	$\mp$	$\int g_{n-1}(x) dx = g_n(x)$
$f^{(n)}(x) = 0$		

$$\therefore \int f(x)g(x)dx = f(x)g_1(x) - f'(x)g_2(x) + f''(x)g_3(x) + \cdots \mp f^{(n-1)}(x)g_n(x) + c$$

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Ex.(4): Find the following integrals:

1- $I = \int x^2 e^x \, dx$

Let $f(x) = x^2$ and $g(x) = e^x$

$f(x)$ and its derivatives		$g(x)$ and its integrals
x^2	+	e^x
$2x$	−	e^x
2	+	e^x
0		e^x

$\therefore I = x^2e^x - 2xe^x + 2e^x + c = e^x(x^2 - 2x + 2) + c$

2- $I = \int (x^3 - 2x^2 + 3x + 1) \sin 2x \, dx$

$f(x)$ and its derivatives		$g(x)$ and its integrals
$x^3 - 2x^2 + 3x + 1$	+	$\sin 2x$
$3x^2 - 2x + 3$	−	$\frac{-1}{2}\cos 2x$
$6x - 2$	+	$\frac{-1}{4}\sin 2x$
6	−	$\frac{1}{8}\cos 2x$
0		$\frac{1}{16}\sin 2x$

$$I = \frac{-1}{2}(x^3 - 2x^2 + 3x + 1)\cos 2x + \frac{1}{4}(3x^2 - 2x + 3)\sin 2x$$

$$+ \frac{1}{8}(6x - 2)\cos 2x - \frac{6}{16}\sin 2x + c$$

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Ex.(5): Find $I = \int e^x \cdot \sin x \, dx$

*

$u = e^x \quad ; \quad dv = \sin x \, dx$

$du = e^x \, dx \quad ; \quad v = -\cos x$

$$I = -e^x \cos x + \int e^x \cos x \, dx = -e^x \cos x + I_1$$

where $I_1 = \int e^x \cos x \, dx$

$u = e^x \quad ; \quad dv = \cos x \, dx$

$du = e^x \, dx \quad ; \quad v = \sin x \, dx$

$$I_1 = e^x \sin x - \int \sin x \cdot e^x \, dx = e^x \cdot \sin x - I$$

$$\therefore I = -e^x \cos x + e^x \sin x - I \Rightarrow 2I = e^x \sin x - e^x \cos x$$

$$\therefore I = \frac{1}{2} e^x [\sin x - \cos x] + C$$

- Method [7]: « Integration of rational functions »:

- Defn.: A rational function is a quotient of two Polynomials, written as:

$$R(x)=\frac{P_n(x)}{Q_m(x)} \quad , Q_m(x)\neq 0$$

, where $P_n(x)$ and $Q_m(x)$ are polynomials of degree n and m respectively.

If $n>m$, we perform a long division until we obtain a rational function whose numerator degree less than or equal to the denominator degree.
for example:

$$I=\int \frac{x^5-6x^4-2x^2-3x+4}{x^3+2x+3}\,dx$$

$$\begin{array}{r|l} & x^2-6x-2 \\ \hline x^3+2x+3 & x^5-6x^4-2x^2-3x+4 \\ & \mp x^5 \mp 2x^3 \mp 3x^2 \\ \hline & -6x^4-2x^3-5x^2-3x+4 \\ & \pm 6x^4 \qquad \pm 12x^2 \pm 18x \\ \hline & -2x^3+7x^2+15x+4 \\ & \pm 2x^3 \qquad \pm 4x \pm 6 \\ \hline & 7x^2+19x+10 \end{array}$$

$$\therefore I=\int\left[x^2-6x-2+\frac{7x^2+19x+10}{x^3+2x+3}\right]dx$$

$$=\frac{1}{3}x^3-3x^2-2x+\int\frac{7x^2+19x+10}{x^3+2x+3}\,dx$$

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If $n \leq m$, we shall discuss the general method of the seperating $\frac{P_n(x)}{Q_m(x)}$ into a sum of partial functions.

Case (1): If the m factors of $Q_m(x)$ are all different and simple, that is $Q_m(x) = (x-a_1)(x-a_2)\cdots(x-a_m)$. then we assign the sum of m partial fractions to these factors as follows

$$\frac{A_1}{(x-a_1)} + \frac{A_2}{(x-a_2)} + \cdots + \frac{A_m}{(x-a_m)}$$, where $A_1, A_2, \ldots, A_m$ are constants must be evaluated

---

Ex. (1): Find $I = \int \frac{x^2+3x+3}{x^3-x}\,dx$

$$I = \int \frac{x^2+3x+3}{x(x-1)(x+1)}\,dx$$

$$\frac{x^2+3x+3}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{(x-1)} + \frac{C}{(x+1)} = \frac{A(x-1)(x+1) + Bx(x+1) + Cx(x-1)}{x(x-1)(x+1)}$$

$\therefore\ x^2+3x+3 = A(x-1)(x+1) + Bx(x+1) + Cx(x-1)$

at $x=0 \Rightarrow 3 = A(-1)(1) + 0 + 0 \Rightarrow A = -3$

at $x=1 \Rightarrow 7 = 0 + B(1)(1+1) + 0 \Rightarrow B = \frac{7}{2}$

at $x=-1 \Rightarrow 1 = 0 + 0 + C(-1)(-1-1) \Rightarrow C = \frac{1}{2}$

or

$$x^2+3x+3 = Ax^2 - A + Bx^2 + Bx + Cx^2 - Cx$$
$$= (A+B+C)x^2 + (B-C)x - A$$

$A+B+C=1$; $B-C=3$; $-A=3 \Rightarrow$

$A=-3$; $B=\frac{7}{2}$; $C=\frac{1}{2}$

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$$\therefore I=\int\left[\frac{-3}{x}+\frac{7/2}{x-1}+\frac{1/2}{x+1}\right]dx=-3\ln x+\frac{7}{2}\ln(x-1)+\frac{1}{2}\ln(x+1)+c$$

Case (2): Repeated factors of $Q_m(x)$:

Suppose $(x-a)^r$ is the highest power of $(x-a)$ which divides $Q_m(x)$. then to this factor we assign the sum of r partial fractions as follows:

$$\frac{A_1}{x-a}+\frac{A_2}{(x-a)^2}+\cdots+\frac{A_r}{(x-a)^r}$$, where $A_1, A_2, \ldots, A_r$ are constants must be found.

for example:

$$\frac{x^3-3x^2+4x-2}{x(x-1)^2(x+1)(x+3)^3}=\frac{A}{x}+\frac{B}{(x-1)}+\frac{C}{(x-1)^2}+\frac{D}{x+1}+\frac{E}{x+3}+\frac{F}{(x+3)^2}+\frac{G}{(x+3)^3}$$

Ex. (2): Evaluate $I=\int\frac{(x+5)\,dx}{(x+2)(x-1)^2}$

$$\frac{x+5}{(x+2)(x-1)^2}=\frac{A}{x+2}+\frac{B}{x-1}+\frac{C}{(x-1)^2}$$

$$=\frac{A(x-1)^2+B(x-1)(x+2)+C(x+2)}{(x+2)(x-1)^2}$$

$$\therefore x+5=A(x-1)^2+B(x-1)(x+2)+C(x+2)$$

at $x=1 \Rightarrow 6=0+0+3C \Rightarrow C=2$

at $x=-2 \Rightarrow 3=9A+0+0 \Rightarrow A=\frac{1}{3}$

at $x=0 \Rightarrow 5=\frac{1}{3}-2B+4 \Rightarrow B=\frac{-1}{3}$

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$$\therefore I = \int \left[\frac{1/3}{x+2} - \frac{1/3}{x-1} + \frac{2}{(x-1)^2} \right] dx$$

$$= \frac{1}{3} \ln(x+2) - \frac{1}{3} \ln(x-1) - \frac{2}{(x-1)} + c$$

$\therefore$ Case (3): Quadratic factors of $Q_m(x)$:

Suppose $(x^2+ax+b)^r$ is the highest power of (x^2+ax+b) which divides $Q_m(x)$. then to this factors we assign the sum of r partial fractions as follows:

$$\frac{A_1x+B_1}{x^2+ax+b} + \frac{A_2x+B_2}{(x^2+ax+b)^2} + \cdots + \frac{A_rx+B_r}{(x^2+ax+b)^r} \quad \text{where}$$

$A_1, A_2, \ldots, A_r, B_1, B_2, \ldots, B_r$ are constant must be found.

for example:

$$\frac{x^2+2x-5}{x^2(x-1)(x^2+1)(x^2+2x+2)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{Dx+E}{x^2+1}$$

$$+ \frac{Fx+G}{x^2+2x+2} + \frac{Hx+I}{(x^2+2x+2)^2}$$

Ex. (3): Find $I = \int \frac{x^5 - x^4 - 3x + 5}{x^4 - 2x^3 + 2x^2 - 2x + 1} dx$

$$\begin{array}{r|l} & x + 1 \\ \hline x^4 - 2x^3 + 2x^2 - 2x + 1 & \cancel{x^5} - x^4 - 3x + 5 \\ & \mp \cancel{x^5} \pm 2x^4 \mp 2x^3 \pm 2x^2 \mp x \\ \hline & \cancel{x^4} - \cancel{2x^3} + \cancel{2x^2} - 4x + 5 \\ & \mp \cancel{x^4} \pm \cancel{2x^3} \mp \cancel{2x^2} \pm 2x \mp 1 \\ \hline & -2x + 4 \end{array}$$

$$I=\int\left[x+1+\frac{-2x+4}{x^4-2x^3+2x^2-2x+1}\right]dx$$

$$=\frac{x^2}{2}+x+\int\frac{(-2x+4)\,dx}{x^4-2x^3+2x^2-2x+1}$$

$$I_1=\int\frac{-2x+4}{(x-1)^2(x^2+1)}\,dx=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{Cx+D}{x^2+1}$$

$$-2x+4=A(x-1)^2(x^2+1)+B(x^2+1)+(Cx+D)(x-1)^2$$

at $x=1 \Rightarrow 2=0+2B+0 \Rightarrow B=1$

at $x=0 \Rightarrow 4=A(-1)(1)+B(1)+D(-1)^2 \Rightarrow$

$4=-A+B+D \Rightarrow 3=-A+D$

at $x=-1 \Rightarrow 6=A(-2)(2)+2B+(-C+D)(4) \Rightarrow$

$1=-A-C+D$

at $x=2 \Rightarrow 0=A(1)(5)+B(5)+(2C+D)(1) \Rightarrow$

$-5=5A+2C+D$

$\Rightarrow A=-2,\ B=1,\ C=2,\ D=1$

$$\therefore I_1=\int\left[\frac{-2}{x-1}+\frac{1}{(x-1)^2}+\frac{2x+1}{x^2+1}\right]dx$$

$$=-2\ln(x-1)+\frac{1}{x-1}+\int\frac{2x\,dx}{x^2+1}+\int\frac{dx}{x^2+1}$$

$$\therefore I=\frac{x^2}{2}+x-2\ln(x-1)-\frac{1}{x-1}+\ln(x^2+1)+\tan^{-1}x+C$$

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- Method [8]: « Integration of Irrational Functions »:

- Case (I): If the integral contain a single irrational expression of the form $\sqrt[q]{ax+b} = (ax+b)^{1/q}$

let $z = (ax+b)^{1/q} \Rightarrow z^q = ax+b \Rightarrow q\, z^{q-1} = a\, dx$

$\Rightarrow dx = \frac{q}{a} z^{q-1} dz$

Ex. (1): Evaluate $I = \int \frac{2x+3}{\sqrt{x+2}} dx$

☆

let $z = (x+2)^{1/2} \Rightarrow z^2 = x+2 \Rightarrow 2z\, dz = dx$

$\therefore I = \int \frac{2(z^2-2)+3}{z} 2z\, dz = 2\int (2z^2-1)\, dz$

$= 2\left[\frac{2}{3}z^3 - z\right] + c = 2\left[\frac{2}{3}(x+2)^{3/2} - (x+2)^{1/2}\right] + c$

Ex. (2): Find $I = \int \frac{\sqrt[3]{x+1}}{x} dx$

☆

let $z = (x+1)^{1/3} \Rightarrow z^3 = x+1 \Rightarrow 3z^2 dz = dx$

$I = \int \frac{z \cdot 3z^2 dz}{z^3-1} = 3\int \frac{z^3 dz}{z^3-1} = 3\int \left[1 + \frac{1}{z^3-1}\right] dz$

$= 3z + \int \frac{3\, dz}{z^3-1}$

let $I_1 = \int \frac{3\, dz}{z^3-1} = \int \frac{3\, dz}{(z-1)(z^2+z+1)}$

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$$\frac{3}{(z-1)(z^2+z+1)} = \frac{A}{(z-1)} + \frac{Bz+C}{z^2+z+1}$$

$$3 = A(z^2+z+1) + (Bz+C)(z-1)$$

at $z=1 \Rightarrow 3 = 3A \Rightarrow A=1$

at $z=0 \Rightarrow 3 = 1-C \Rightarrow C=-2$

at $z=-1 \Rightarrow 3 = 1+(-B-2)(-2) = 1+2B+4 \Rightarrow B=-1$

$$\therefore I_1 = \int \left[\frac{1}{z-1} - \frac{z+2}{z^2+z+1}\right] dz$$

$$= \ln(z-1) - \frac{1}{2}\int \frac{2z+1-1}{z^2+z+1} dz - \int \frac{2\,dz}{z^2+z+1}$$

$$= \ln(z-1) - \frac{1}{2}\int \frac{2z+1}{z^2+z+1} dz + \frac{1}{2}\int \frac{dz}{z^2+z+1} - \int \frac{2\,dz}{z^2+z+1}$$

$$= \ln(z-1) - \frac{1}{2}\ln(z^2+z+1) - \frac{3}{2}\int \frac{dz}{(z+\frac{1}{2})^2+\frac{3}{4}}$$

$$= \ln(z-1) - \frac{1}{2}\ln(z^2+z+1) - \frac{3}{2}\cdot\frac{2}{\sqrt{3}}\tan^{-1}\left[\frac{z+1/2}{\sqrt{3}/2}\right] + C$$

$$\therefore I = 3(x+1)^{1/3} + \ln\left[(x+1)^{1/3}-1\right] - \frac{1}{2}\ln\left[(x+1)^{2/3}+(x+1)^{1/3}+1\right]$$

$$-\sqrt{3}\tan^{-1}\left[\frac{2(x+1)^{1/3}}{\sqrt{3}}\right] + C$$

Ex.(3): Evaluate $I = \int \frac{dx}{\sqrt[3]{x^2}+\sqrt{x}}$

$\Rightarrow\star$ let $z = x^{\frac{1}{6}} \Rightarrow z^6 = x \Rightarrow 6z^5\,dz = dx$

$$\therefore I = \int \frac{6z^5\,dz}{z^4+z^3} = 6\int \frac{z^5\,dz}{z^3(z+1)} = 6\int \frac{z^2\,dz}{z+1}$$

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$$I = 6\int\left[z - 1 + \frac{1}{z+1}\right]dz = 6\left[\frac{z^2}{2} - z + \ln(z+1)\right] + c$$

$$= 6\left[\frac{1}{2}x^{1/3} - x^{1/6} + \ln(x^{1/6}+1)\right] + c$$

Ex.(4): Evaluate $I = \int \frac{\sqrt{x}\,dx}{1+\sqrt[4]{x}}$

☆

Let $z = x^{1/4} \Rightarrow z^4 = x \Rightarrow 4z^3 dz = dx$

$$I = \int \frac{z^2 \cdot 4z^3 dz}{1+z} = 4\int \frac{z^5\,dz}{1+z} = 4\int\left[z^4 - z^3 + z^2 - z + 1 - \frac{1}{z+1}\right]dz$$

$$= 4\left[\frac{z^5}{5} - \frac{z^4}{4} + \frac{z^3}{3} - \frac{z^2}{2} + z - \ln(z+1)\right] + c$$

$$= 4\left[\frac{1}{5}x^{5/4} - \frac{1}{4}x + \frac{1}{3}x^{3/4} - \frac{1}{2}x^{1/2} + x^{1/4} - \ln(x^{1/4}+1)\right] + c$$

- **Case (II):** If a single irrational expression of the form $\sqrt{a^2-x^2}$ or $\sqrt{a^2+x^2}$ or $\sqrt{x^2-a^2}$, the z substitution to the radical reduces the given integral to that of the rational form

---

Ex.(5): Evaluate $I = \int \frac{\sqrt{4-x^2}}{x^3}\,dx$

☆

$$I = \int \frac{(4-x^2)^{1/2}}{x^3}\,dx$$

Let $z = (4-x^2)^{1/2} \Rightarrow z^2 = 4-x^2 \Rightarrow x^2 = 4-z^2 \Rightarrow$

$$2x\,dx = -2z\,dz \Rightarrow x\,dx = -z\,dz$$

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$$I=\int \frac{(4-x^2)^{1/2}}{x^3}\,dx=\int \frac{(4-x^2)^{1/2}}{x^4}\,x\,dx=\int \frac{z\cdot(-z)\,dz}{(4-z^2)^2}$$

$$=-\int \frac{z^2\,dz}{(z+2)^2(z-2)^2}$$

$$\frac{-z^2}{(z+2)^2(z-2)^2}=\frac{A}{z+2}+\frac{B}{(z+2)^2}+\frac{C}{z-2}+\frac{D}{(z-2)^2}$$

$$A=\frac{1}{8}\ ;\ B=\frac{-1}{4}\ ;\ C=\frac{-1}{8}\ ;\ D=\frac{-1}{4}$$

$$\therefore I=\int\left[\frac{1/8}{z+2}-\frac{1/4}{(z+2)^2}-\frac{1/8}{z-2}-\frac{1/4}{(z-2)^2}\right]dz$$

$$=\frac{1}{8}\ln(z+2)+\frac{1}{4}\frac{1}{z+2}-\frac{1}{8}\ln(z-2)+\frac{1}{4}\frac{1}{z-2}+C$$

$$=\frac{1}{8}\ln\left[\frac{z+2}{z-2}\right]+\frac{1}{4}\left[\frac{1}{z+2}+\frac{1}{z-2}\right]+C$$

$$=\frac{1}{8}\ln\left[\frac{z+2}{z-2}\right]+\frac{1}{4}\left[\frac{2z}{z^2-4}\right]+C$$

$$=\frac{1}{8}\ln\left[\frac{\sqrt{4-x^2}+2}{\sqrt{4-x^2}-2}\right]-\frac{1}{2}\left[\frac{\sqrt{4-x^2}}{x^2}\right]+C$$

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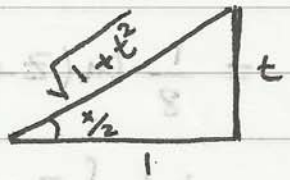
- Method [9]: « Integration of Rational Functions of Trigonometrics »

- If the integrand is a rational function of trigonometric, the substitution of $t = \tan\frac{x}{2}$ will reduce the integrand to a rational function of t which can be handle by Method [7].

we have:

$$t = \tan\frac{x}{2} \Rightarrow \frac{x}{2} = \tan^{-1} t \Rightarrow \frac{dx}{2} = \frac{dt}{1+t^2} \Rightarrow$$

$$dx = \frac{2\,dt}{1+t^2}$$



$$\sin\frac{x}{2} = \frac{t}{\sqrt{1+t^2}} \quad ; \quad \cos\frac{x}{2} = \frac{1}{\sqrt{1+t^2}}$$

$$\sin x = 2\sin\frac{x}{2}\cdot\cos\frac{x}{2} = \frac{2t}{1+t^2}$$

$$\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \frac{1-t^2}{1+t^2}$$

Ex.(1): Find $I = \int \frac{dx}{5-4\cos x}$

$$I = \int \frac{\frac{2\,dt}{1+t^2}}{5-4\frac{1-t^2}{1+t^2}} = \int \frac{2\,dt}{5(1+t^2)-4(1-t^2)} = 2\int \frac{dt}{1+9t^2}$$

$$= \frac{2}{3}\int \frac{3\,dt}{1+(3t)^2} = \frac{2}{3}\tan^{-1} 3t + c = \frac{2}{3}\tan^{-1}\left[3\tan\frac{x}{2}\right] + c$$

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Ex.(2): Evaluate $I=\int \frac{dx}{3\cos x+4\sin x}$

$$I=\int \frac{\frac{2\,dt}{1+t^2}}{3\left[\frac{1-t^2}{1+t^2}\right]+4\left[\frac{2t}{1+t^2}\right]}=2\int \frac{dt}{3(1-t^2)+8t}$$

$$=2\int \frac{dt}{3-3t^2+8t}=-2\int \frac{dt}{3t^2-8t-3}=-2\int \frac{dt}{(3t+1)(t-3)}$$

$$\frac{1}{(3t+1)(t-3)}=\frac{A}{(3t+1)}+\frac{B}{t-3}$$

$$1=A(t-3)+B(3t+1)\Rightarrow$$

$$A=\frac{-3}{10}\quad;\quad B=\frac{1}{10}$$

$$\therefore I=-2\int\left[\frac{-3/10}{3t+1}+\frac{1/10}{t-3}\right]dt=\frac{1}{5}\left[\int\frac{3\,dt}{3t+1}-\int\frac{dt}{t-3}\right]$$

$$=\frac{1}{5}\ln(3t+1)-\frac{1}{5}\ln(t-3)+c$$

$$=\frac{1}{5}\ln\left[\frac{3t+1}{t-3}\right]+c=\frac{1}{5}\ln\left[\frac{3\tan\frac{x}{2}+1}{\tan\frac{x}{2}-3}\right]+c$$

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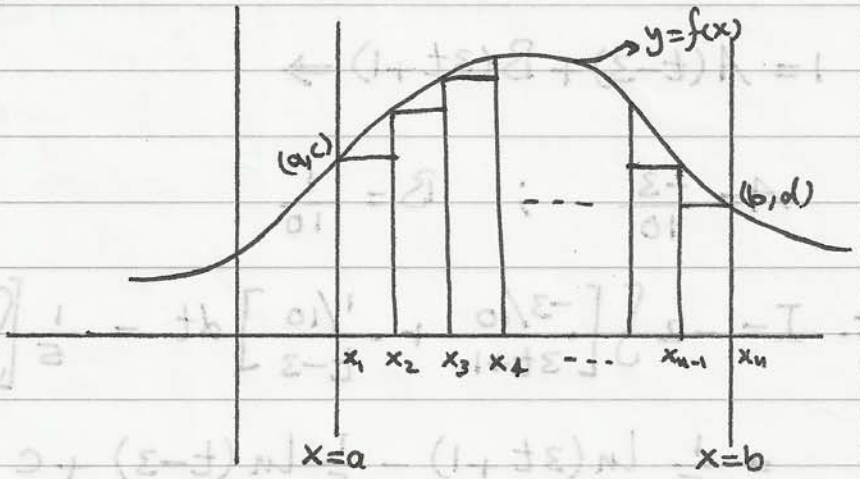
«Applications of Definite Integrals»

[1] The Area under the Curve:

- Approximating with Rectangle Areas:

- Let $y=f(x)$ be a continuous function of x on the close interval $[a,b]$ and suppose we wish to find the area bounded by the graph of f, the lines $x=a$ and $x=b$, and the x-axis.



Let A_a^b be the required area.

we divide the area A_a^b into n thin strips of uniform width Δx by lines parallel to the y-axis.

Area of 1st rectangle $= f(x_1)\,\Delta x = f(a)\,\Delta x$

Area of 2nd rectangle $= f(x_2)\,\Delta x$

Area of 3rd rectangle $= f(x_3)\,\Delta x$

$\vdots$

Area of nth rectangle $= f(x_n)\,\Delta x = f(b)\,\Delta x$

Then the area A_a^b is approximately:

$$\text{Area} = A_a^b \approx f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x$$

$$= \sum_{i=1}^{n} f(x_i)\Delta x \qquad \cdots (1)$$

- **Defn.:** The area under the graph of a continuous function f over an interval $[a,b]$ is the limit of the sums of areas of inscribed rectangles of equal base Δx as their number n increases without bound, that is:

$$\left.\begin{aligned} A_a^b &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \\ A_a^b &= \lim_{\Delta x\to 0} \sum_{x_i=a}^{b} f(x_i)\Delta x \end{aligned}\right\} \cdots (2)$$

In 19th century Georg Friedrich Bernhard Riemann (1826-1866) put on a solid logical foundation and a famous theorem as a consequence of existance the limit in eq.(2). His idea was to trap the limit between upper and lower sums and he call this limit "The Riemann Integral" of f over the interval $[a,b]$ and denoted by $\int_a^b f(x)\,dx$.

- The Integral Existence theorem: "Existence of Riemann Integral"

- If f is Continuous on $[a,b]$, then:

$$\lim_{\text{norm}\to 0} \sum_{x_i=a}^{b} f(x_i) = \int_a^b f(x)\,dx \quad \text{--- (3)}$$

So, by the first foundamental theorem of integral calculus:

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a) \quad \text{--- (4)}$$

Accordingly, the area A_a^b is:

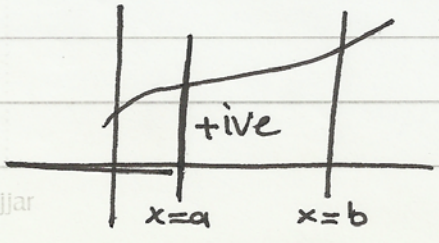
$$A_a^b = \int_a^b f(x)\,dx \quad \text{--- (5)}$$

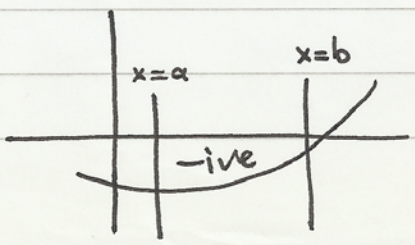
- Notes:

1- If x Can be expressed as a function of y, Say $x = g(y)$, then:

$$A_a^b = \int_c^d g(y)\,dy$$

2- If a Continuous function f is +ive over $[a,b]$, then the area under its graph is +ive and if f is −ive over $[a,b]$ then the area under its graph is −ive.





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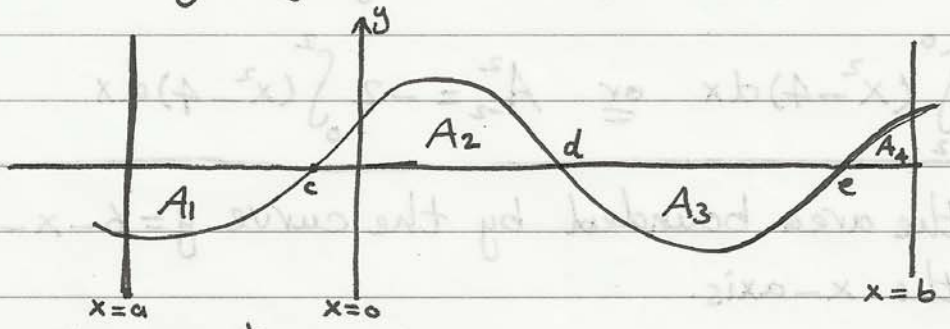
3- If the graph of $y=f(x)$ over $[a,b]$ is partly below and partly above the x-axis as show in the following figure

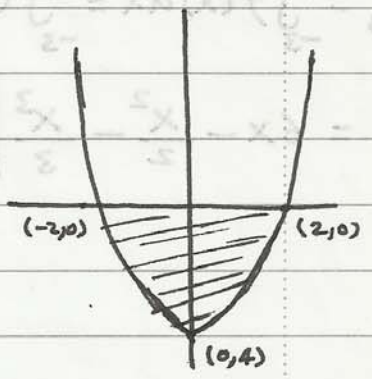


$$A_a^b = -A_1 + A_2 - A_3 + A_4$$

Then

$$A_a^b = \int_a^b f(x)dx = -\int_a^c f(x)dx + \int_c^d f(x)dx - \int_d^e f(x)dx + \int_e^b f(x)dx$$

Ex.(1): Find the area bounded by the curve $y=x^2-4$ and the x-axis.



$y = x^2-4$

$x^2-4=0 \Rightarrow x=\mp 2$

$$A_a^b = \int_a^b f(x)dx$$

$$A_{-2}^2 = \int_{-2}^2 (x^2-4)dx = \frac{x^3}{3} - 4x\Big|_{-2}^2$$

$$= \left(\frac{8}{3}-8\right) - \left(\frac{-8}{3}+8\right) = \frac{16}{3} - 16 = 16\left[\frac{1}{3}-1\right] = 16\left(\frac{-2}{3}\right)$$

$$= \frac{-32}{3} \text{ below the x-axis}$$

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In fact $A_{-2}^{2} = -\int_{-2}^{2} (x^2-4)\,dx = \frac{32}{3}$ unit^2

⁂ because of the symmetry, one may calculate the area as:

$$A_{-2}^{2} = -2\int_{-2}^{0} (x^2-4)\,dx \quad \text{or} \quad A_{-2}^{2} = -2\int_{0}^{2} (x^2-4)\,dx$$

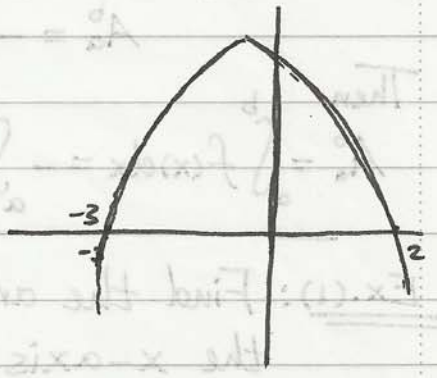
Ex.(2): Find the area bounded by the curve $y = 6-x-x^2$ and the x-axis.

⁂

$y = 6-x-x^2$

$6-x-x^2 = 0 \Rightarrow (3+x)(2-x) = 0$

$\Rightarrow x = -3, 2$



$$A_{-3}^{2} = \int_{-3}^{2} f(x)\,dx = \int_{-3}^{2} (6-x-x^2)\,dx$$

$$= 6x - \frac{x^2}{2} - \frac{x^3}{3}\Big|_{-3}^{2} = \left[12 - 2 - \frac{8}{3}\right] - \left[-18 - \frac{9}{2} + 9\right]$$

$$= 10 - \frac{8}{3} + 9 + \frac{9}{2} = 19 - \frac{8}{3} + \frac{9}{2}$$

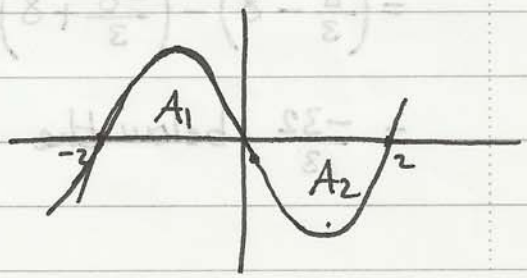
$$= \frac{195}{6} \text{ unit}^2$$

Ex.(3): Find the total area bounded by the curve $y = x^3 - 4x$ and the x-axis

⁂

$x^3 - 4x = 0 \Rightarrow x(x-2)(x+2) = 0$

$\Rightarrow x = 0, 2, -2$



$$A_a^b = \int_a^b f(x)\,dx$$

$$A_{-2}^{2} = \int_{-2}^{2} (x^3 - 4x)\,dx = \frac{x^4}{4} - 2x^2 \Big|_{-2}^{2} = (4-8) - (4-8) = 0$$

((wrong answer))

$$A_{-2}^{2} = A_1 - A_2 = \int_{-2}^{0} f(x)dx - \int_{0}^{2} f(x)dx$$

$$= \int_{-2}^{0} (x^3 - 4x)\,dx - \int_{0}^{2} (x^3 - 4x)\,dx = \left[\frac{x^4}{4} - 2x^2\right]_{-2}^{0} - \left[\frac{x^4}{4} - 2x^2\right]_{0}^{2}$$

$$= \left[(0-0) - (4-8)\right] - \left[(4-8) - (0-0)\right] = 4 + 4 = 8 \text{ unit}^2$$

or one may find the solution as follows:

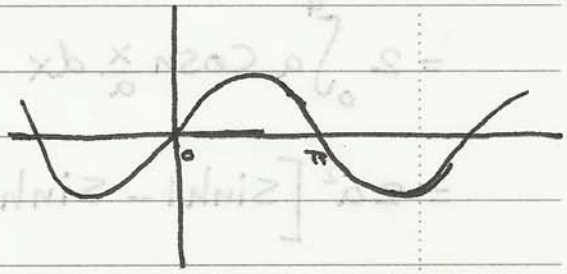
$$A_{-2}^{2} = 2A_1 = 2\int_{-2}^{0} f(x)dx = 2\int_{-2}^{0} (x^3 - 4x)dx = 2\left[\frac{x^4}{4} - 2x^2\right]_{0}^{-2} = 8$$

or

$$A_{-2}^{2} = -2A_2 = -2\int_{0}^{2} f(x)dx = -2\int_{0}^{2} (x^3 - 4x)\,dx = -2\left[\frac{x^4}{4} - 2x^2\right]_{0}^{2} = 8$$

Ex. (4): Find the area bounded by one arch of the curve $y = \sin x$ and the x-axis

$$A_a^b = \int_a^b f(x)\,dx$$



$$A_0^{\pi} = \int_0^{\pi} \sin x\,dx = -\cos x \Big|_0^{\pi}$$

$$= -(\cos\pi - \cos 0) = -(-1-1) = 2 \text{ unit}^2$$

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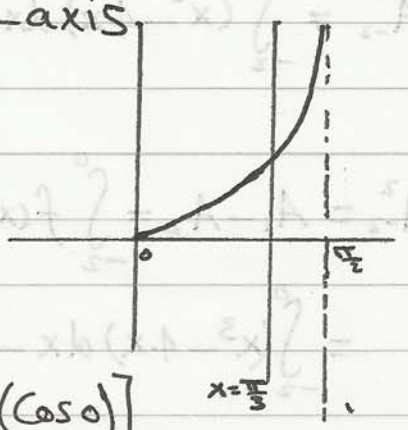
Ex.(5): Find the area of the region in the first quadrant bounded by the curve $y=\tan x$, the line $x=\frac{\pi}{3}$ and the x-axis.



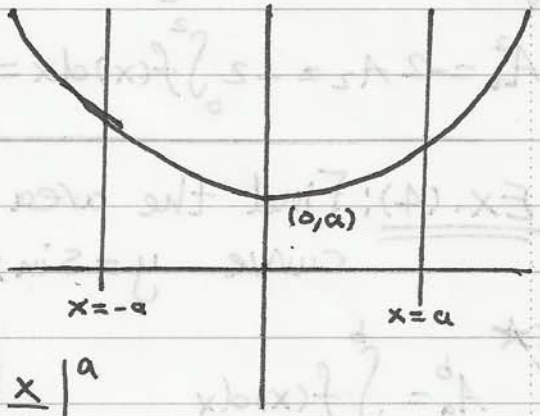
$$A_a^b = \int_a^b f(x)\,dx$$

$$A_0^{\pi/3} = \int_0^{\pi/3} \tan x\,dx = \int_0^{\frac{\pi}{3}} \frac{\sin x}{\cos x}\,dx$$

$$= -\ln(\cos x)\Big|_0^{\pi/3} = -\left[\ln\left(\cos\tfrac{\pi}{3}\right) - \ln(\cos 0)\right]$$

$$= -\left[\ln\tfrac{1}{2} - \ln 1\right] = -\ln\tfrac{1}{2} - \ln 0 = \ln 2$$

Ex.(6): Find the area bounded by the curve $y=\frac{a}{2}\left[e^{\frac{x}{a}}+e^{\frac{-x}{a}}\right]$, the lines $x=-a$, $x=a$ and x-axis.



$$A_a^b = \int_a^b f(x)\,dx$$

$$A_{-a}^{a} = \int_{-a}^{a} f(x)\,dx = 2\int_0^2 f(x)\,dx$$

$$= 2\int_0^a a\cosh\frac{x}{a}\,dx = 2a^2 \sinh\frac{x}{a}\Big|_0^a$$

$$= 2a^2\left[\sinh 1 - \sinh 0\right] = 2a^2\,\frac{e^2-1}{2e} = \frac{a^2(e^2-1)}{e}$$

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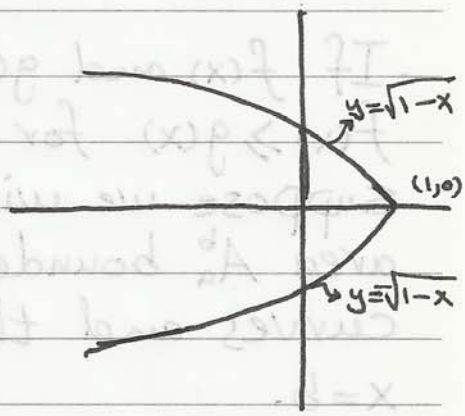
Ex.(7): Find the area between the curve $x = 1 - y^2$ and the y-axis



$1 - y^2 = 0 \Rightarrow y = \mp 1$

$$A_a^b = A_c^d = \int_c^d g(y)\,dy$$

$$A_{-1}^{1} = \int_{-1}^{1} (1-y^2)\,dy = y - \frac{y^3}{3}\Big|_{-1}^{1}$$

$$= \left(1 - \frac{1}{3}\right) - \left(-1 + \frac{1}{3}\right) = \frac{4}{3} \text{ unit}^2$$

or

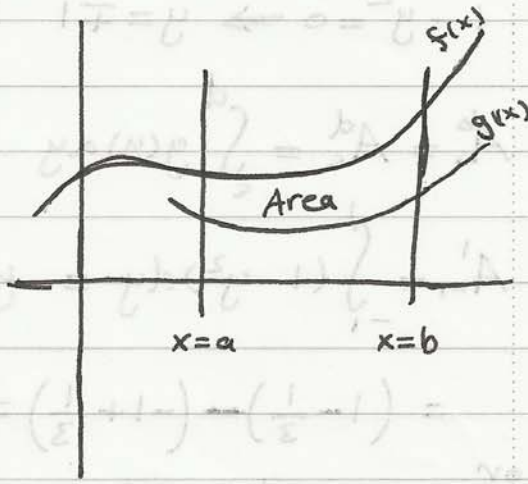
$$A_{-1}^{1} = 2\int_0^1 (1-y^2)\,dy = \frac{4}{3}$$

$$A_a^b = \int_a^b f(x)\,dx \Rightarrow A_0^1 = 2\int_0^1 f(x)\,dx = 2\int_0^1 \sqrt{1-x}\,dx$$

$$= -2\,\frac{(1-x)^{3/2}}{3/2}\Big|_0^1 = \frac{-4}{3}[0-1] = \frac{4}{3}$$

- Area Between Two Curves:

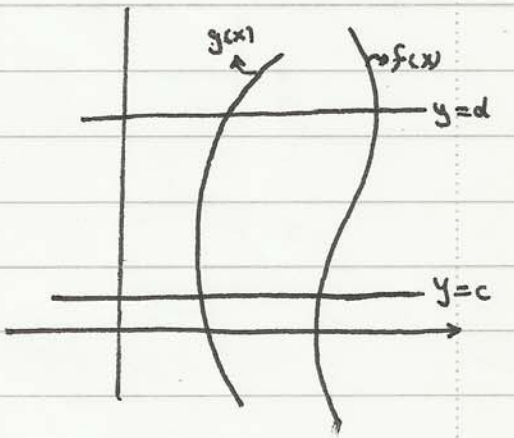
- If $f(x)$ and $g(x)$ are continuous functions with $f(x) \geq g(x)$ for $a \leq x \leq b$, and suppose we wish to find the area A_a^b bounded by the two curves and the lines $x=a$, $x=b$.



The area under the curve $f(x)$ for $a \leq x \leq b$ is $\int_a^b f(x)dx$, and the area under the curve $g(x)$ for $a \leq x \leq b$ is $\int_a^b g(x)dx$. then the required area is:

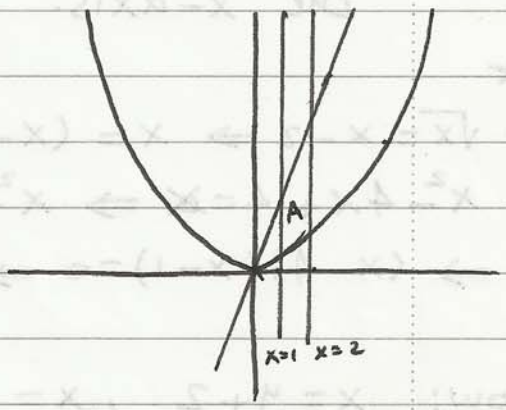
$$A_a^b = \int_a^b f(x)dx - \int_a^b g(x)dx = \int_a^b [f(x)-g(x)]dx$$

Similarly, if $f(y)$ and $g(y)$ are continuous functions with $f(y) \geq g(y)$ for $c \leq y \leq d$, then the area bounded by their curves and the linens $y=c$, $y=d$ is:



$$A_c^d = \int_c^d [f(y)-g(y)]dy$$

Ex.(1): Find the area bounded by $y=3x$, $y=x^2$, $x=1$, $x=2$

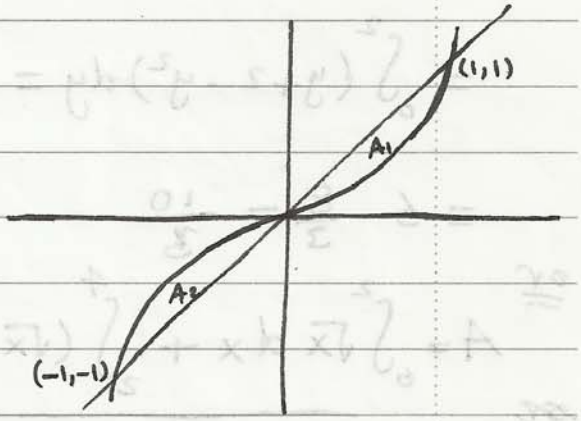


$$A_a^b = \int_a^b [f(x)-g(x)]\,dx$$

$$A_1^2 = \int_1^2 [3x-x^2]\,dx$$

$$= \frac{3}{2}x^2 - \frac{x^3}{3}\Big|_1^2 = \frac{13}{6}$$

Ex.(2): Find the area bounded by $y=x$ and $y=x^3$.



$$x^3=x \Rightarrow x^3-x=0 \Rightarrow$$

$$x(x-1)(x+1)=0 \Rightarrow x=0,-1,1$$

$$A_a^b = \int_a^b [f(x)-g(x)]\,dx$$

$$A = A_1+A_2 = 2A_1 = 2A_2$$

$$A_{-1}^1 = 2\int_0^1 [x-x^3]\,dx = 2\left[\frac{x^2}{2}-\frac{x^4}{4}\right]_0^1 = 2\left[\frac{1}{2}-\frac{1}{4}\right] = \frac{1}{2}$$

or

$$A_{-1}^1 = 2\int_{-1}^0 [x^3-x]\,dx = \frac{1}{2}$$

or

$$A_{-1}^1 = \int_{-1}^0 (x^3-x)\,dx + \int_0^1 (x-x^3)\,dx = \frac{1}{4}+\frac{1}{4} = \frac{1}{2}$$

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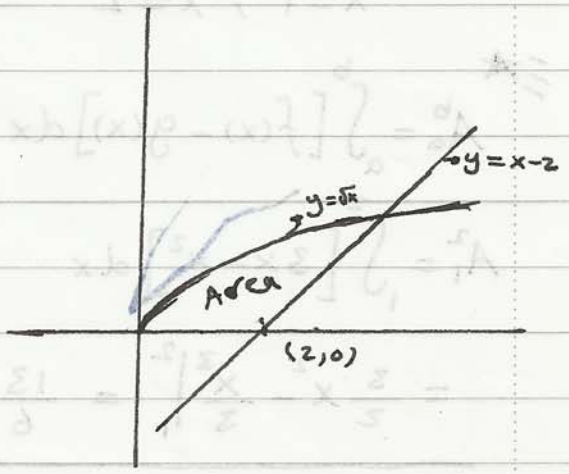
Ex.(3): Find the area bounded by $y = x-2$, $y = \sqrt{x}$ and the x-axis.



$\sqrt{x} = x-2 \Rightarrow x = (x-2)^2 \Rightarrow$

$x^2 - 4x + 4 = x \Rightarrow x^2 - 5x + 4 = 0$

$\Rightarrow (x-4)(x-1) = 0 \Rightarrow x = 4, 1$

now: $x = y+2$, $x = y^2$

$$A_c^d = \int_c^d [f(y) - g(y)]\,dy$$

$$= \int_0^2 (y+2-y^2)\,dy = \frac{y^2}{2} + 2y - \frac{y^3}{3}\Big|_0^2 = 2 + 4 - \frac{8}{3}$$

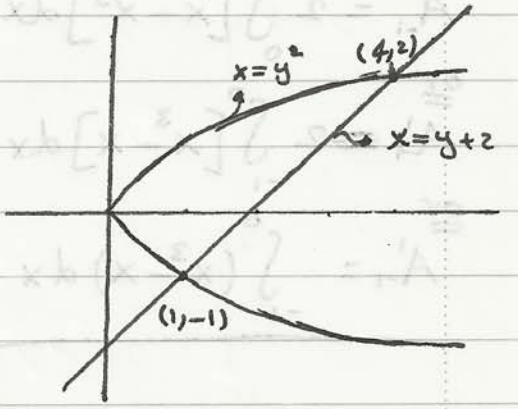
$$= 6 - \frac{8}{3} = \frac{10}{3}$$

or

$$A = \int_0^2 \sqrt{x}\,dx + \int_2^4 (\sqrt{x} - x + 2)\,dx = \frac{10}{3}$$

or

Ex.(4): Find the area bounded by $y^2 = x$ and $y = x-2$



$y^2 = y+2 \Rightarrow y^2 - y - 2 = 0 \Rightarrow$

$(y-2)(y+1) = 0 \Rightarrow y = 2, -1$

$x = 4, 1$

$$A_c^d = \int_{-1}^{2} (y+2-y^2)\,dy = \frac{y^2}{2} + 2y - \frac{y^3}{3}\Big|_{-1}^{2}$$

$$= \left(2 + 4 - \frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right) = \frac{9}{2}$$

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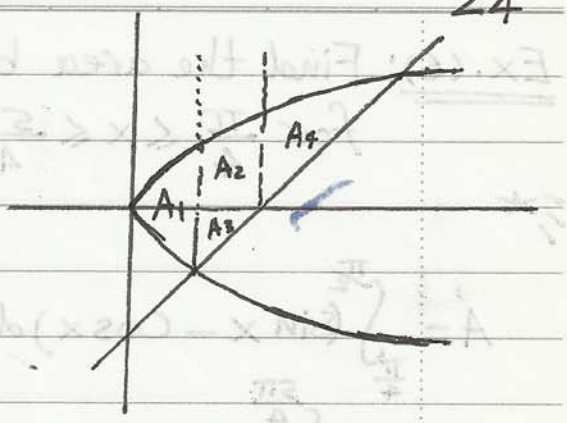


or

$$A_a^b = A_1 + A_2 + A_3 + A_4$$

$$A_1 = 2\int_0^1 \sqrt{x} = \frac{4}{3}$$

$$A_2 = \int_1^2 \sqrt{x}\,dx = \frac{4\sqrt{2}-2}{3}$$

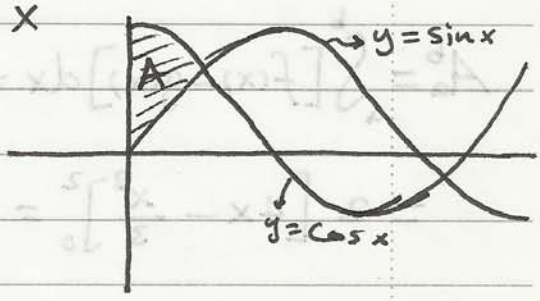
$$A_3 = -\int_1^2 (x-2)\,dx = \frac{1}{2}$$

$$A_4 = \int_2^4 (\sqrt{x} - x + 2)\,dx = \frac{10-4\sqrt{2}}{3}$$

$$\therefore A = \frac{4}{3} + \frac{4\sqrt{2}-2}{3} + \frac{1}{2} + \frac{10-4\sqrt{2}}{3} = \frac{4-4\sqrt{2}-2+10-4\sqrt{2}}{3} + \frac{1}{2}$$

$$= \frac{12}{3} + \frac{1}{2} = 4 + \frac{1}{2} = \frac{9}{2}$$

Ex.(5): Find the area of the region in the first quadrant bounded by the y-axis and the curves $y = \sin x$, $y = \cos x$



$$\sin x = \cos x \Rightarrow \tan x = 1 \Rightarrow$$

$$x = \frac{\pi}{4} \Rightarrow y = \frac{1}{\sqrt{2}}$$

$$A_a^b = \int_a^b [f(x) - g(x)]\,dx$$

$$A_0^{\pi/4} = \int_0^{\pi/4} [\cos x - \sin x]\,dx = \sin x + \cos x \Big|_0^{\frac{\pi}{4}} = \left[\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right] - (0+1)$$

$$= \sqrt{2} - 1$$

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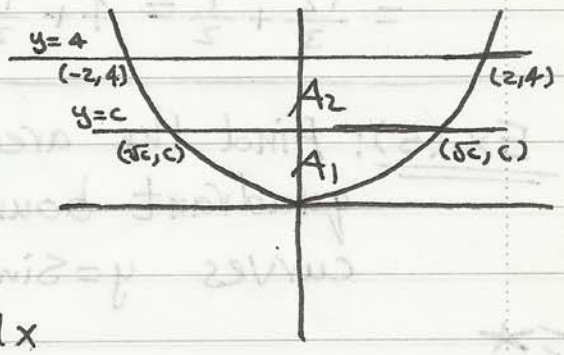
Ex.(6): Find the area between $y=\sin x$ and $y=\cos x$ for $\frac{\pi}{4} \le x \le \frac{5\pi}{4}$

$$A = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\sin x - \cos x)\,dx + \int_{\frac{\pi}{2}}^{\pi} \sin x\,dx - \int_{\frac{\pi}{2}}^{\pi} \cos x\,dx - \int_{\pi}^{\frac{5\pi}{4}} (\cos x - \sin x)\,dx$$

$$= \left[-\cos x - \sin x\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} + \left[-\cos x\right]_{\frac{\pi}{2}}^{\pi} - \left[\sin x\right]_{\frac{\pi}{2}}^{\pi} - \left[\sin x + \cos x\right]_{\pi}^{\frac{5\pi}{4}}$$

$$= 2\sqrt{2}$$

Ex.(7): The area bounded by the curve $y=x^2$ and the line $y=4$ is divided into two equal partitions by the line $y=c$. Find c.



Total area $\equiv A = A_1 + A_2$

$$= 2A_1 = 2A_2$$

$$A_a^b = \int_a^b [f(x) - g(x)]\,dx = 2\int_0^2 (4 - x^2)\,dx$$

$$= 2\left[4x - \frac{x^3}{3}\right]_0^2 = \frac{32}{3}$$

$$\frac{A}{2} = A_1 \Rightarrow \frac{32/3}{2} = 2\int_0^{\sqrt{c}} (c - x^2)\,dx = 2\left[cx - \frac{x^3}{3}\right]_0^{\sqrt{c}} \Rightarrow$$

$$\frac{8}{3} = c\sqrt{c} - \frac{c\sqrt{c}}{3} = c\sqrt{c}\left(1 - \frac{1}{3}\right) = \frac{2c\sqrt{c}}{3} \Rightarrow c\sqrt{c} = 4 \Rightarrow$$

$$c^{3/2} = 4 \Rightarrow c = 4^{2/3} = \sqrt[3]{16}.$$

[2] Volume of a Solid of Revolution:

- The solid generated by rotating a plane region about an axis in its plane is called a solid of revolution.

- **Defn.:** The volume of a solid of known cross-sectional area $A(x)$ from $x=a$ to $x=b$ is

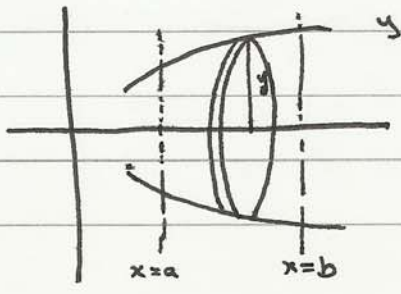
$$\text{Volume} \equiv V = \int_a^b A(x)\,dx$$

In practice, the steps to calculate Volumes of solids are:

1- Draw the region whose volume is to be found.
2- Find $A(x)$.
3- Find the limits of integration.
4- Integrate.

(i) Disk method: «Washer»

Let $y=f(x)$ be a cont. on close interval $[a,b]$ and suppose that the region bounded by $f(x)$ and the lines $x=a$, $x=b$ and the x-axis is revolved about the x-axis



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Imagine the solid to be cut into thin slices of thickness Δx by planes perpendicular to the axis of rotation, we see that each slice is a disk of area $A(x) = \pi y^2$ and volume $\Delta V = A(x)\,\Delta x$.
When the number of slices increase indefinitly, then the total volume V of the solid can be defined as the limit sum of these volumes, that is:

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i)\,\Delta x_i = \lim_{\Delta x\to 0} \sum_{x=a}^{b} A(x)\,\Delta x = \int_a^b A(x)\,dx$$

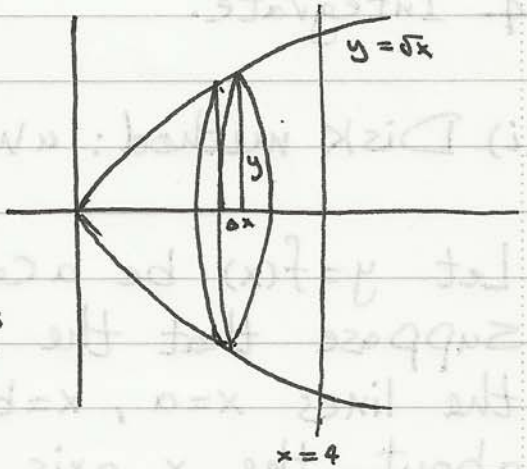
$$= \int_a^b \pi y^2\,dx = \pi \int_a^b [f(x)]^2\,dx$$

Ex.(1): Find the volume of the solid generated when the area bounded by the curve $y = \sqrt{x}$ and the lines $x=0$, $x=4$ and the x-axis is revolved about (i) x-axis. (ii) y-axis.

(i) Area $\equiv A = \pi y^2$

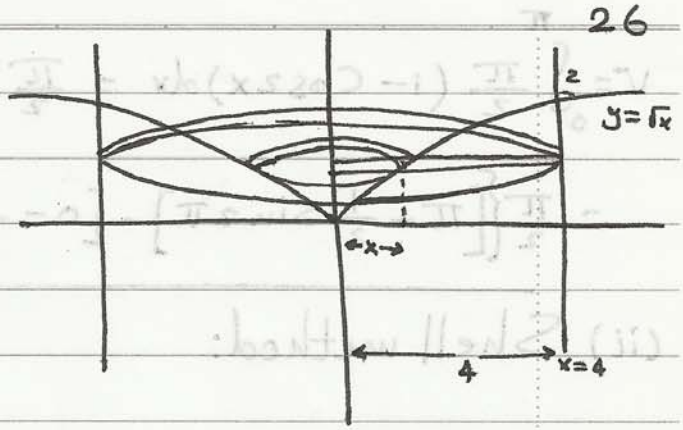
$$V = \int_a^b A\,dx = \int_0^4 \pi y^2\,dx = \pi \int_0^4 x\,dx$$

$$= \frac{\pi}{2} x^2 \Big|_0^4 = \frac{\pi}{2}(16-0) = 8\pi \text{ unit}^3$$



(ii) Area $= \pi(4)^2 - \pi(x^2)$

$= \pi(16 - x^2)$

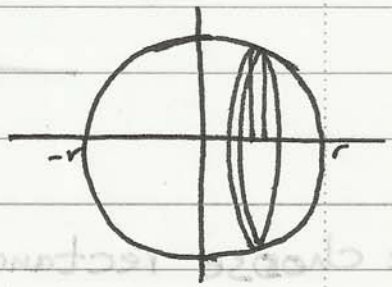


$$V = \int_c^d A\,dy$$

$$= \int_0^2 \pi(16 - x^2)\,dy$$

$$= \pi \int_0^2 (16 - y^4)\,dy = \pi\left[16y - \frac{y^5}{5}\right] = \pi\left[32 - \frac{32}{5}\right] = \frac{128\pi}{5}$$

Ex.(2): The region inside the circle $x^2+y^2=r^2$ is rotated about the axis to generate a solid sphere. Find its Volume.



$A = \pi y^2$

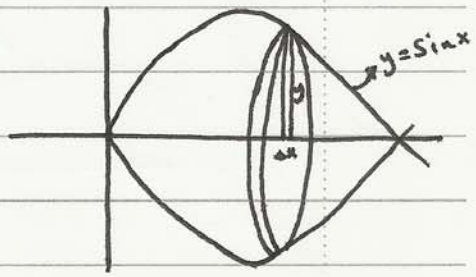
$$V = \int_{-r}^{r} A\,dx = \int_{-r}^{r} \pi y^2\,dx$$

$$= 2\pi \int_0^r (r^2 - x^2)\,dx = 2\pi\left[r^2x - \frac{x^3}{3}\right]_0^r$$

$$= 2\pi\left[r^3 - \frac{r^3}{3}\right] = 2\pi r^3\left[1 - \frac{1}{3}\right] = \frac{4}{3}\pi r^3 \text{ unit}^3$$

Ex. (3): Find the volume of the solid generated by revolving the region bounded by the x-axis and one arch of the curve $y = \sin x$ about the x-axis

Area $\equiv A = \pi y^2$



$$V = \int_0^{\pi} \pi y^2\,dx = \int_0^{\pi} \pi \sin^2 x\,dx$$

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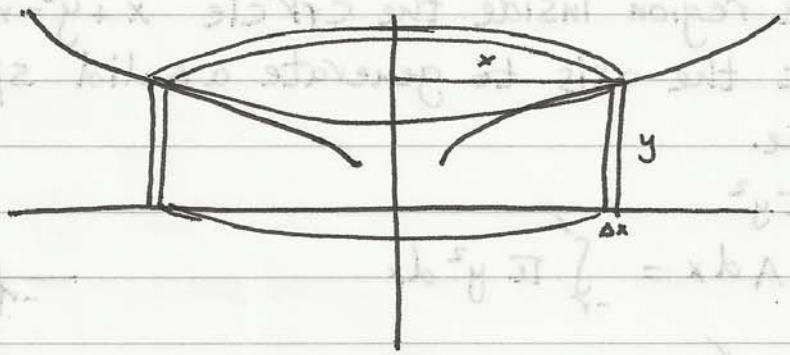
$$V = \int_0^{\pi} \frac{\pi}{2}(1-\cos 2x)\,dx = \frac{\pi}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^{\pi}$$

$$= \frac{\pi}{2}\left\{\left[\pi - \frac{1}{2}\sin 2\pi\right] - \left[0 - \frac{1}{2}\sin 0\right]\right\} = \frac{\pi^2}{2} \text{ unit}^3$$

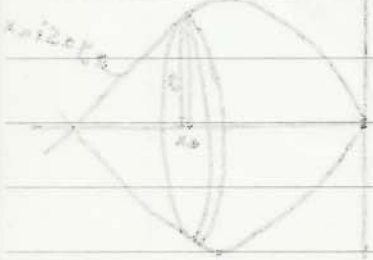
(ii) Shell method:

- Let $y = f(x)$ be a cont. func. on $[a,b]$ and suppose that the region bounded by $f(x)$ and the lines $x=a$, $x=b$ and the x-axis is revolved about the y-axis



We choose rectangle strips of thickness dx that are parallel to the axis of rotation and therefore sweep out cylindrical shells instead of disks each of area $A(x) = 2\pi xy$ and volume $dV = A(x)\,dx$. then the total volume is:

$$V = \int_a^b A(x)\,dx = \int_a^b 2\pi xy\,dx = 2\pi \int_a^b x\,f(x)\,dx$$



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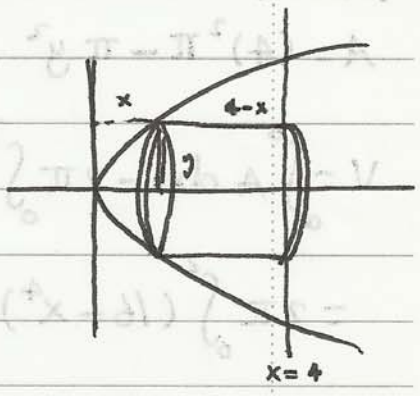
Ex.(4): Solve Ex.(1) using shell method.



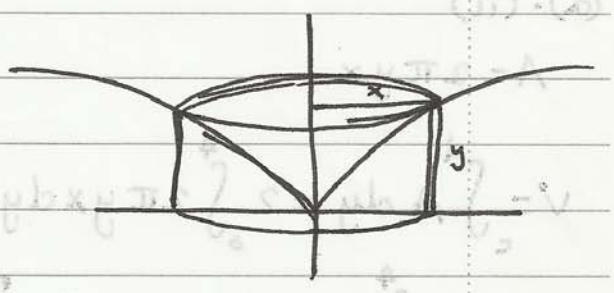
(i) $A = 2\pi y(4-x)$

$$V = \int_c^d A\,dy = \int_0^2 2\pi y(4-x)\,dx$$

$$= 2\pi \int_0^2 y(4-y^2)\,dy = 2\pi \left[2y^2 - \frac{y^4}{4}\right]_0^2$$

$$= 2\pi \left[8-4\right] = 8\pi \quad \text{unit}^3$$

(ii)



$A = 2\pi xy$

$$V = \int_a^b A\,dx = 2\pi \int_0^4 xy\,dx$$

$$= 2\pi \int_0^4 x\sqrt{x}\,dx = 2\pi \left.\frac{x^{5/2}}{5/2}\right|_0^4 = \frac{4}{5}\pi\left[4^{5/2} - 0\right] = \frac{128}{5}\pi$$

Ex.(5): The area bounded by the curve $y = x^2$ and the line $y = 4$ generate various solids of revolution when rotated about:

(a) x-axis, (b) y-axis, (c) line $y = 4$, (d) $y = -1$, (e) $x = 2$.

Find the volume generated in each case by using (i) disk method. (ii) shell method.

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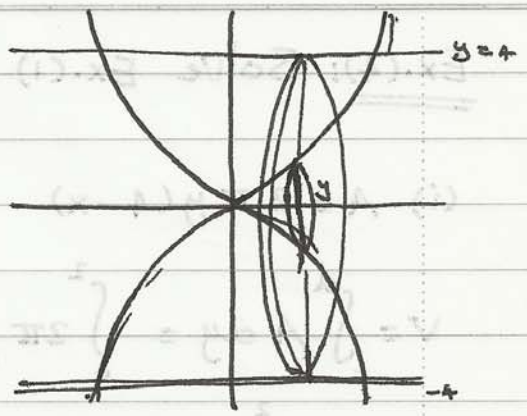
(a). (i)



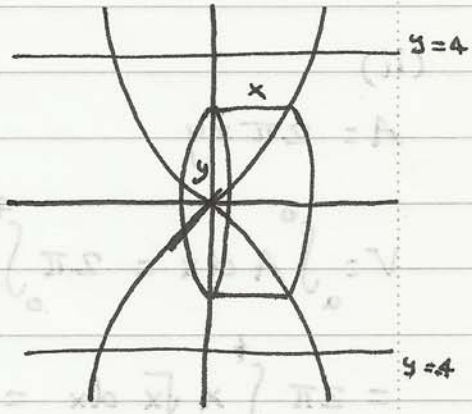
$$A = (4)^2\pi - \pi y^2$$

$$V = \int_a^b A\,dx = 2\pi \int_0^2 (16 - y^2)\,dx$$

$$= 2\pi \int_0^2 (16 - x^4)\,dx$$

$$= 2\pi \left[16x - \frac{x^5}{5}\right]_0^2 = 2\pi \left[32 - \frac{32}{5}\right] = \frac{256\pi}{5} \text{ unit}^3$$

(a). (ii)



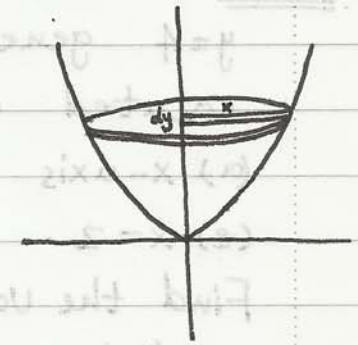
$$A = 2\pi y x$$

$$V = \int_c^d A\,dy = 2\int_0^4 2\pi y x\,dy$$

$$= 4\pi \int_0^4 y\sqrt{y}\,dy = 4\pi \left[\frac{y^{5/2}}{5/2}\right]_0^4$$

$$= \frac{8\pi}{5}\left(4^{5/2}\right) = \frac{256\pi}{5} \text{ unit}^3$$

(b) (i)



$$A = \pi x^2$$

$$V = \int_c^d A\,dy = \int_0^4 \pi x^2\,dy = \pi \int_0^4 y\,dy$$

$$= \pi \left[\frac{y^2}{2}\right]_0^4 = \frac{\pi}{2}(16 - 0) = 8\pi \text{ unit}^3$$

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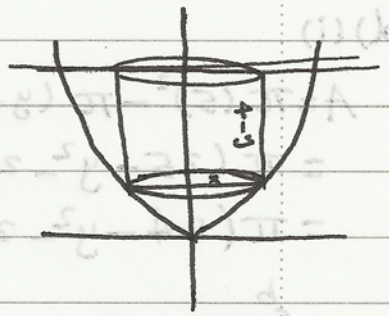
(b)(ii)



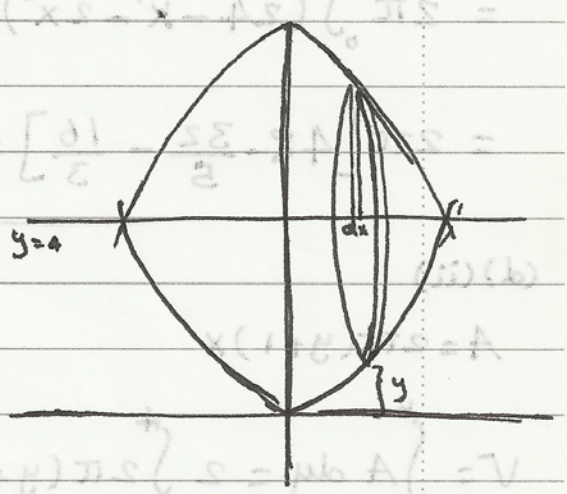
$A = 2\pi x(4-y)$

$$V = \int_a^b A\,dx = 2\pi \int_0^2 x(4-y)\,dx$$

$$= 2\pi \int_0^2 x(4-x^2)\,dx = 2\pi \int_0^2 (4x - x^3)\,dx$$

$$= 2\pi \left[2x^2 - \frac{x^4}{4}\right]_0^2 = 2\pi(8-4) = 8\pi \text{ unit}^3$$

(C)(i)



$A = \pi(4-y)^2$

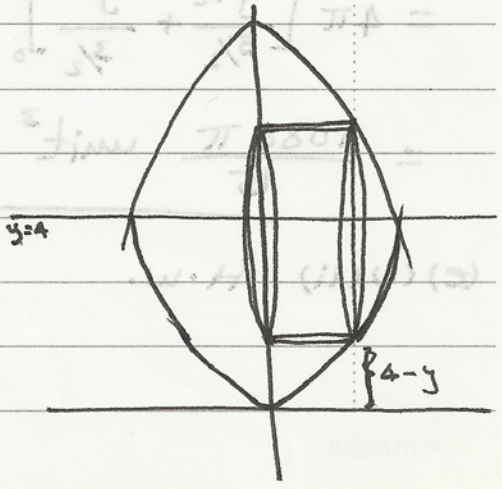
$$V = \int_a^b A\,dx = 2\int_0^2 \pi(4-y^2)^2\,dx$$

$$= 2\pi \int_0^2 (4-x^2)^2\,dx$$

$$= 2\pi \int_0^2 (16 - 8x^2 + x^4)\,dx$$

$$= 2\pi \left[16x - \frac{8}{3}x^3 + \frac{x^5}{5}\right]_0^2 = 2\pi\left[32 - \frac{64}{3} + \frac{32}{5}\right] = \frac{512\pi}{15}$$

(c)(ii)



$A = 2\pi(4-y)x$

$$V = \int_a^b A\,dy = 2\int_0^4 2\pi(4-y)x\,dy$$

$$= 4\pi \int_0^4 (4-y)\sqrt{y}\,dy = \frac{512\pi}{15}$$

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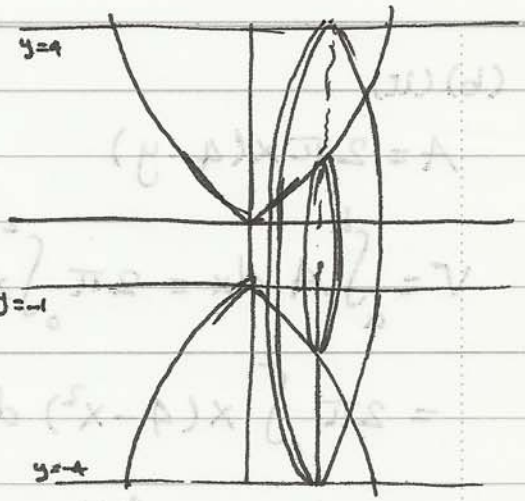
(d)(i)



$$A = \pi(5)^2 - \pi(y+1)^2$$

$$= \pi(25 - y^2 - 2y - 1)$$

$$= \pi(24 - y^2 - 2y)$$

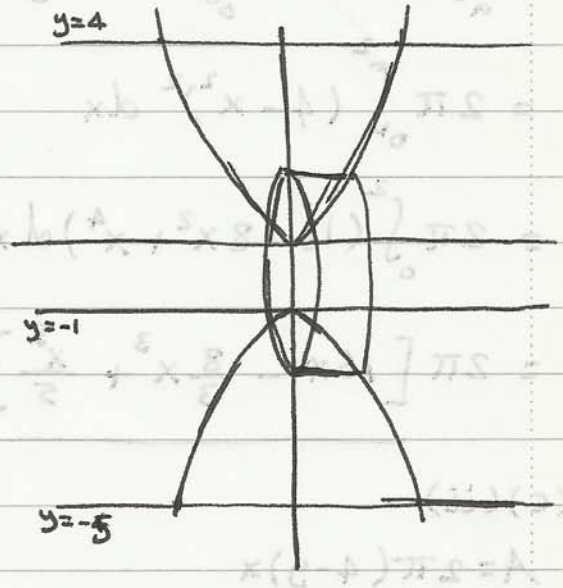
$$V = \int_a^b A\,dx$$

$$= 2\int_0^2 \pi(24 - y^2 - 2y)\,dx$$

$$= 2\pi\int_0^2 (24 - x^4 - 2x^2)\,dx = 2\pi\left[24x - \frac{x^5}{5} - \frac{2}{3}x^3\right]_0^2$$

$$= 2\pi\left[48 - \frac{32}{5} - \frac{16}{3}\right] = \frac{1088\pi}{15} \text{ unit}^3$$

(d)(ii)



$$A = 2\pi(y+1)x$$

$$V = \int_a^b A\,dy = 2\int_0^4 2\pi(y+1)x\,dy$$

$$= 4\pi\int_0^4 (y+4)\sqrt{y}\,dy$$

$$= 4\pi\left[\frac{y^{5/2}}{5/2} + \frac{y^{3/2}}{3/2}\right]_0^4$$

$$= \frac{1088\pi}{15} \text{ unit}^3$$

(e)(i)(ii) H.w.

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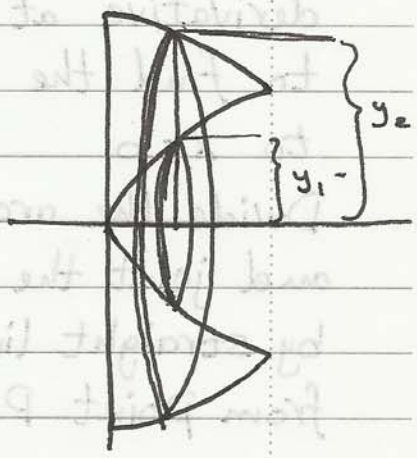
Ex.(6): The region bounded by $y=0$ and the curves $y=\cos x$ and $y=\sin x$ for $0 \le x \le \frac{\pi}{4}$ is revolved about the x-axis. Find the volume of the generated solid.



$$A = \pi y_1^2 - \pi y_2^2 = \pi(\cos^2 x - \sin^2 x)$$
$$= \pi \cos 2x$$

$$V = \int_a^b A\,dx = \int_0^{\frac{\pi}{4}} \pi \cos 2x\,dx$$

$$= \frac{\pi}{2} \cdot \sin 2x \Big|_0^{\frac{\pi}{4}} = \frac{\pi}{2}(1-0) = \frac{\pi}{2} \text{ unit}^3$$

Ex.(7): Repeat Ex.(6) by revolving about the y-axis.

Ans.: $\pi\left(\frac{\pi}{\sqrt{2}} - 1\right)$

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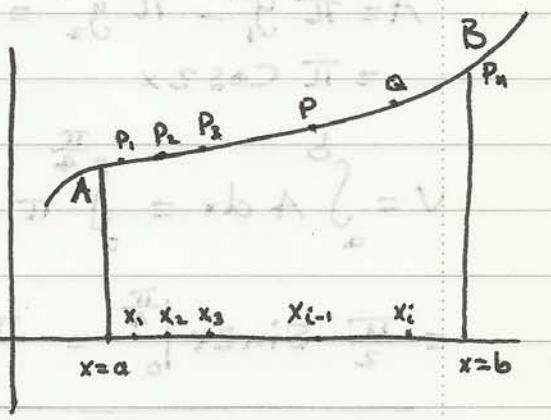
[3] Length of A plane Curve «Arc length»:

Let $y=f(x)$ be a cont. func. over $[a,b]$, and has a cont. derivative at each point $[a,b]$, and suppose we wish to find the length of the curve $y=f(x)$ from $x=a$ to $x=b$

Divide the arc AB into n pieces and joint the successive points by straight lines. the distance from point P to point Q, is



$$\overline{PQ} = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$$

the length of the curve AB is approximatly

$$L_A^B \approx \sum_{i=1}^{n} \sqrt{(\Delta x_i)^2 + (\Delta y_i)}$$

when the number of division points is increased indefinitely, that $n \longrightarrow \infty$, we see that $\Delta x_i \longrightarrow 0$, where we obtain:

$$L_A^B = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \lim_{\Delta x\to 0} \sum_{a}^{b} \sqrt{(\Delta x)^2 + (\Delta y)^2} \quad \ldots (1)$$

Provided the limit in eq.(1) exist.

Let (x,y) be some point between P and Q where the tangent line is parallel the line $\overline{PQ}$, then:

$f'(x) = \frac{\Delta y}{\Delta x} \Rightarrow \Delta y = f'(x) \cdot \Delta x$, Hence eq.(1) may written as

$$L_A^B = \lim_{\Delta x\to 0} \sum_{a}^{b} \sqrt{(\Delta x)^2 + \{f'(x)\}^2 \cdot (\Delta x)^2} = \lim_{\Delta x\to 0} \sum_{a}^{b} \sqrt{1 + \{f'(x)\}^2} \cdot \Delta x$$

Then according to the integral existance theorem (Riemann):

$$L_A^B = \int_a^b \sqrt{1+\{f'(x)\}^2}\,dx = \int_a^b \sqrt{1+\left[\frac{dy}{dx}\right]^2}\,dx \quad \cdots (2)$$

Notes: (1) If x expressed as a func. of y, say $x=g(y)$, then:

$$L_A^B = \int_c^d \sqrt{1+\{g(y)\}^2}\,dy = \int_c^d \sqrt{1+\left[\frac{dx}{dy}\right]^2}\,dy \quad \cdots (3)$$

(2) If $x=x(t)$, and $y=y(t)$, then:

$$L_A^B = \int_{ta}^{tb} \sqrt{\left[\frac{dx}{dt}\right]^2+\left[\frac{dy}{dt}\right]^2}\,dt \quad \cdots (4)$$

(3) Usually we denote $L_A^B = \int_a^b ds$, where $ds = \sqrt{(dx)^2+(dy)^2}$

Ex.(1): Find the length of the curve $y = \frac{4\sqrt{2}}{3}x^{3/2} - 1$, from $x=0$ to $x=1$

$$L_A^B = \int_a^b \sqrt{1+\left[\frac{dy}{dx}\right]^2}\,dx$$

$$y = \frac{4\sqrt{2}}{3}x^{\frac{3}{2}} - 1 \Rightarrow \frac{dy}{dx} = \frac{4\sqrt{2}}{3}\cdot\frac{3}{2}x^{\frac{1}{2}} = 2\sqrt{2}\,x^{1/2}$$

$$\Rightarrow \left[\frac{dy}{dx}\right]^2 = 8x$$

$$L_A^B = \int_0^1 \sqrt{1+8x}\,dx = \frac{1}{8}\,\frac{(1+8x)^{3/2}}{3/2}\Big|_0^1 = \frac{1}{12}\left(9^{3/2}-1\right) = \frac{13}{6}$$

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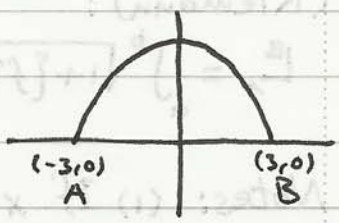
Ex.(2): Find the length of the arc whose eqn. $y = \sqrt{9 - x^2}$

$$L_A^B = \int_a^b \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\, dx$$



$$y = \sqrt{9 - x^2} \Rightarrow \frac{dy}{dx} = \frac{-2x}{2\sqrt{9-x^2}} = \frac{-x}{\sqrt{9-x^2}}$$

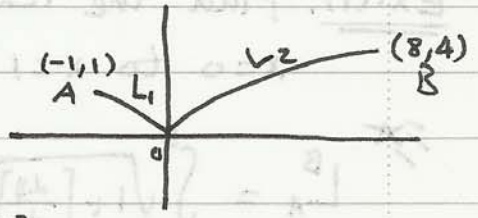
$$\left[\frac{dy}{dx}\right]^2 = \frac{x^2}{9-x^2} \Rightarrow 1 + \left[\frac{dy}{dx}\right]^2 = 1 + \frac{x^2}{9-x^2} = \frac{9}{9-x^2}$$

$$\therefore L_A^B = \int_{-3}^{3} \frac{3}{\sqrt{9-x^2}}\, dx = 2\int_0^3 \frac{3}{\sqrt{9-x^2}}\, dx$$

$$= 6 \sin^{-1}\frac{x}{3}\Big|_0^3 = 6\left[\sin^{-1}(1) - \sin^{-1}(0)\right] = 6\left[\frac{\pi}{2} - 0\right] = 3\pi$$

Ex.(3): Find the length of the curve $y = x^{2/3}$ from $x=-1$ to $x=8$

$$L_A^B = \int_a^b \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\, dx$$



$$y = x^{\frac{2}{3}} \Rightarrow \frac{dy}{dx} = \frac{2}{3}x^{-\frac{1}{3}} \rightarrow$$

$$\left[\frac{dy}{dx}\right]^2 = \frac{4}{9}x^{-\frac{2}{3}} \Rightarrow 1 + \left[\frac{dy}{dx}\right]^2 = 1 + \frac{4}{9}x^{-\frac{2}{3}} = 1 + \frac{4}{9\sqrt[3]{x^2}}$$

$$L_1 = -\int_{-1}^{0} \sqrt{1 + \frac{4}{9}x^{-\frac{2}{3}}}\, dx \quad \text{and} \quad L_2 = \int_0^8 \sqrt{1 + \frac{4}{9}x^{-\frac{2}{3}}}\, dx$$

and $L_A^B = L_1 + L_2$

H.W: Find L_1 and L_2

or

We may write $x = y^{\frac{3}{2}}$, and $L_A^B = \int_c^d \sqrt{1 + \left[\frac{dx}{dy}\right]^2}\, dy$

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$$\frac{dx}{dy} = \frac{3}{2} y^{\frac{1}{2}} \Rightarrow \left[\frac{dx}{dy}\right]^2 = \frac{9}{4} y$$

$$L_1 = \int_0^1 \sqrt{1+\frac{9}{4}y}\, dy = \frac{4}{9} \frac{\left[1+\frac{9}{4}y\right]^{3/2}}{3/2}\Big|_0^1 = \frac{8}{27}\left[\left(1+\frac{9}{4}\right)^{3/2} - 1\right]$$

$$= \frac{8}{27}\left[\frac{13\sqrt{13}}{8} - 1\right] = \frac{1}{27}\left(13\sqrt{13} - 8\right)$$

$$L_2 = \int_0^4 \sqrt{1+\frac{9}{4}y}\, dy = \frac{4}{9} \frac{\left[1+\frac{9}{4}y\right]^{3/2}}{3/2}\Big|_0^4 = \frac{8}{27}\left[10\sqrt{10} - 1\right]$$

$$= \frac{1}{27}\left[80\sqrt{10} - 8\right]$$

$$L_A^B = L_1 + L_2 = \frac{1}{27}\left[13\sqrt{13} - 8 + 80\sqrt{10} - 8\right] = 10.4$$

Ex. (4): Find the length of the curve whose parametric eq.s are: $x = 2(\theta - \sin\theta)$, $y = 2(1-\cos\theta)$ for $0 \le \theta \le 2\pi$

$$L_A^B = \int_{\theta_a}^{\theta_b} \sqrt{\left[\frac{dx}{d\theta}\right]^2 + \left[\frac{dy}{d\theta}\right]^2}\, d\theta$$

$$x = 2(\theta - \sin\theta) \Rightarrow \frac{dx}{d\theta} = 2(1-\cos\theta) \Rightarrow \left[\frac{dx}{d\theta}\right]^2 = 4(1-\cos\theta)^2$$

$$y = 2(1-\cos\theta) \Rightarrow \frac{dy}{d\theta} = 2\sin\theta \Rightarrow \left[\frac{dy}{d\theta}\right]^2 = 4\sin^2\theta$$

$$\left[\frac{dx}{d\theta}\right]^2 + \left[\frac{dy}{d\theta}\right]^2 = 4(1-\cos\theta)^2 + 4\sin^2\theta$$

$$= 4 - 8\cos\theta + 4\cos^2\theta + 4\sin^2\theta$$

$$= 8 - 8\cos\theta = 8\left(2\sin^2\frac{\theta}{2}\right) = 16\sin^2\frac{\theta}{2}$$

$$L_0^{2\pi} = \int_0^{2\pi} \sqrt{16\sin^2\frac{\theta}{2}}\, d\theta = 4\int_0^{2\pi} \sin\frac{\theta}{2}\, d\theta = -8\cos\frac{\theta}{2}\Big|_0^{\pi} = 16$$

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Ex. (5): Find the distance traveled between $t=0$ and $t=\frac{\pi}{2}$ by a particle whose position at time t is given by $x=\sin^2 t$, $y=\cos^2 t$

$$L_A^B = \int_{ta}^{tb} \sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2}\, dt$$

$$x=\sin^2 t \rightarrow \frac{dx}{dt} = 2\sin t \cos t \rightarrow \left[\frac{dx}{dt}\right]^2 = 4\sin^2 t \cos^2 t$$

$$y=\cos^2 t \Rightarrow \frac{dy}{dt} = -2\cos t \cdot \sin t \Rightarrow \left[\frac{dy}{dt}\right]^2 = 4\sin^2 t \cdot \cos^2 t$$

$$\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2 = 4\sin^2 t\cos^2 t + 4\sin^2 t\cos^2 t = 8\sin^2 t\cos^2 t$$

$$L_0^{\pi/2} = \int_0^{\frac{\pi}{2}} \sqrt{8\sin^2 t\cos^2 t}\, dt = 2\sqrt{2}\int_0^{\frac{\pi}{2}} \sin t\cos t\, dt$$

$$= 2\sqrt{2}\ \frac{\sin^2 t}{2}\Big|_0^{\frac{\pi}{2}} = \sqrt{2}\left[\sin^2\frac{\pi}{2} - \sin^2 0\right] = \sqrt{2}$$

Exercises of Method [1]:

(1) $\int (x-\frac{1}{x})^2 dx$

(2) $\int \coth^{-3} 2x \cdot \operatorname{csch}^2 2x \, dx$

(3) $\int \frac{1+e^{2x}}{e^x} dx$

(4) $\int x \, 2^{x^2+3} \, dx$

(5) $\int \frac{\sec^2 x}{1+\tan^2 x} dx$

(6) $\int \frac{\sin\sqrt{x}}{\sqrt{x}} dx$

(7) $\int \frac{e^x \cdot dx}{1+e^{2x}}$

(8) $\int \tan 4x \, dx$

(9) $\int \frac{dx}{x[1+(\ln x)^2]}$

(10) $\int \frac{(\ln x)^3}{x} dx$

(11) $\int \tan^2 3x \cdot dx$

(12) $\int \csc x \cdot dx$

(13) $\int \operatorname{sech} x \cdot dx$

(14) $\int \operatorname{csch} x \cdot dx$

(15) $\int \frac{\sec^2(\ln x)}{x} dx$

(16) $\int \frac{2\sin\sqrt{x}}{\sqrt{x} \cdot \sec\sqrt{x}} \cdot dx$

(17) $\int \tan x \cdot \ln(\cos x) \cdot dx$

(18) $\int \frac{x\sqrt{x}}{1+x^5} dx$

(19) $\int_0^{\frac{\pi}{2}} \sin^3 x \cdot \cos x \cdot dx$

(20) $\int_0^{1/2} \frac{3\sin^{-1} x}{\sqrt{1-x^2}} dx$

(21) $\int_0^1 \frac{\sqrt{1+e^{-2x}}}{e^{-3x}} dx$

(22) $\int_0^{\pi/2} \sin^2 x \, dx$

(23) $\int \frac{x-2}{\sqrt{9-x^2}} dx$

(24) $\int \frac{dx}{\sqrt{x}(1+x)}$

(25) $\int_0^{\infty} e^{-x-e^{-x}} \cdot dx$

(26) Show that: $\int_0^1 f(1-x)dx = \int_0^1 f(x)dx$

- Exercises of Method [2]:

(1) $\int \sin^5 2x \, dx$

(2) $\int \cos^3 x \cdot \sin^{-\frac{1}{2}} x \, dx$

(3) $\int \cot^4 3x \, dx$

(4) $\int \sin^6 x \, dx$

(5) $\int \cot^3 2x \cdot \csc^4 2x \, dx$

(6) $\int \csc^6 x \, dx$

(7) $\int_0^{\pi} \cos^2 x \cdot dx$

(8) $\int_0^{\pi/2} \cos^4 x \, dx$

(9) $\int_{-\pi/4}^{\pi/4} \sin^3 2x \cdot dx$

(10) $\int_0^{\pi/3} \tan^3 x \cdot \sec x \cdot dx$

(11) $\int_0^{1} \sinh^4 x \cdot dx$

(12) $\int x \sin^3 x^2 \, dx$

(13) $\int_0^{\pi/2} \sin^2 x \cdot \cos^2 x \cdot dx$

(14) $\int \tan \frac{x}{3} \cdot \sec^2 \frac{x}{3} \, dx$

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Exercises of Method [3]:

(1) $\int_{-6}^{-2\sqrt{3}} \frac{dx}{x\sqrt{x^2-9}}$

(2) $\int_{0}^{2} \frac{x^2\,dx}{x^2+4}$

(3) $\int_{0}^{2\sqrt{3}} \frac{x^3\,dx}{\sqrt{x^2+4}}$

(4) $\int \frac{\sqrt{4x^2-9}}{x}\,dx$

(5) $\int_{0}^{\sqrt{5}} x^2\sqrt{5-x^2}\,.\,dx$

(6) $\int_{1}^{3} \frac{dx}{x^4\sqrt{x^2+3}}$

(7) $\int \frac{dx}{(9-x^2)^{3/2}}$

(8) $\int_{3}^{3\sqrt{3}} \frac{dx}{x^2\sqrt{9+x^2}}$

(9) $\int_{0}^{1} \frac{x^2\,dx}{(4-x^2)^{3/2}}$

(10) $\int \frac{dx}{(x^2+4)^2}$

(11) $\int \frac{\sin\theta}{\sqrt{2-\cos^2\theta}}\,d\theta$

(12) $\int \frac{dx}{x^4\sqrt{4-x^2}}$

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Exercises of Method [4]:

(1) $\int_0^3 \frac{dx}{\sqrt{x^2+9}}$

(2) $\int_2^3 \sqrt{x^2-4}\, dx$

(3) $\int \frac{x^2\, dx}{(9+x^2)^{1/2}}$

(4) $\int_2^4 \frac{\sqrt{x^2-4}}{x^2}\, dx$

(5) $\int (3+x^2)^{3/2}\, dx$

(6) $\int x^2 (5+x^2)^{1/2}\, dx$

(7) $\int \frac{x^3\, dx}{(5-x^2)^{3/2}}$

(8) $\int x^2 \sqrt{5-x^2}\, dx$

(9) $\int \csc x\, dx$

(10) $\int \operatorname{sech} x\, dx$

(11) $\int \sec^3 x\, dx$

(12) $\int \csc^3 x\, dx$

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Exercises of Method [5]:

(1) $\int_{1}^{2} \frac{dx}{x^2+2x+5}$

(2) $\int_{1}^{2} \frac{3\,dx}{9x^2-6x+5}$

(3) $\int_{-1}^{0} \frac{dx}{\sqrt{3-2x-x^2}}$

(4) $\int \frac{dx}{(x-1)\sqrt{x^2-2x-3}}$

(5) $\int \frac{x+3}{x^2+2x+5}\,dx$

(6) $\int \frac{(2x-3)\,dx}{(x^2+2x-3)^{3/2}}$

(7) $\int \frac{(4x+5)\,dx}{(-x^2+3x-4)^2}$

(8) $\int \frac{(3x+4)\,dx}{\sqrt{x^2+2x}}$

(9) $\int \frac{\sqrt{x^2+2x}}{x+1}\,dx$

(10) $\int \frac{\cos x\,dx}{\sin^2 x+2\sin x+5}$

(11) $\int \frac{(2x-5)\,dx}{\sqrt{4x-x^2}}$

(12) $\lim_{a\to 5} \int_{a}^{-4} \frac{dx}{\sqrt{-x^2-8x-15}}$

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Exercises of Method [6]:

(1) $\int x^2 \ln(x+1)\,dx$

(2) $\int_1^2 x \sec^{-1} x\,dx$

(3) $\int x \tan^{-1} x\,dx$

(4) $\int x (\ln x)^2\,dx$

(5) $\int x^{-2} \tan^{-1} x\,dx$

(6) $\int \sin\sqrt{2x}\,dx$

(7) $\int (x^{-2} + x^{-1} + 1) \ln x\,dx$

(8) $\int (x^3 + x^2 + x + 1) e^{-2x}\,dx$

(9) $\int (x^2 + x - 2) \cos 4x\,dx$

(10) $\int e^{-x} \sin x\,dx$

(11) $\int \sin 2x \sin 5x\,dx$

(12) $\int \sin 2x \sinh 2x\,dx$

(13) $\int \sinh 2x \cdot \cosh 2x\,dx$

(14) $\int \sin(\ln x)\,dx$

(15) $\int_0^1 x\sqrt{1-x}\,dx$

(16) $\int \sqrt{1-x^2} \sin^{-1} x\,dx$

(17) $\int_0^\infty x^3 e^{-x}\,dx$

(18) $\int \ln(x + \sqrt{1+x^2})\,dx$

Exercises of Method [7]:

(1) $\int \frac{x^2+3x+4}{x-2}\,dx$

(2) $\int \frac{x^3-x^2+2x+2}{x^2+3x+2}\,dx$

(3) $\int \frac{x^4+1}{x^3-x}\,dx$

(4) $\int \frac{x^2+2x+3}{(x+1)(x-1)(x-2)}\,dx$

(5) $\int \frac{x^3+3}{x^2+4}\,dx$

(6) $\int \frac{(x^2+2)\,dx}{(x+1)(x-1)^2}$

(7) $\int \frac{x^2+3x+3}{(x+1)(x^2+1)}\,dx$

(8) $\int \frac{x^3+3x^2-2x+1}{x^4+5x^2+4}\,dx$

(9) $\int_{\frac{\pi}{3}}^{\pi/2} \frac{\sin\theta\,d\theta}{\cos^2\theta+\cos\theta-2}$

(10) $\int \frac{x^3-2x^2+3x-4}{(x-1)^2(x^2+2x+2)}\,dx$

(11) $\int \frac{2x^3+3x^2+x-1}{(x^2+2x+2)^2}\,dx$

(12) $\int \frac{x^3-2x-5}{x^2(x-1)^2(x+1)^2}\,dx$

(13) $\int \frac{dx}{x^2(x^2+1)^2}$

(14) $\int_0^{\sqrt{3}} \frac{5x^2}{x^2+1}\,dx$

(15) $\int_0^{\ln 2} \frac{e^x\,dx}{e^{2x}+3e^x+2}$

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Exercises of Method [8]:

(1) $\int \frac{\sqrt{x}+2}{\sqrt{x}-1}\,dx$

(2) $\int x\sqrt{x-1}\,dx$

(3) $\int \frac{2x+1}{(x+2)^{2/3}}\,dx$

(4) $\int x^2(2x+1)^{-1/3}\,dx$

(5) $\int \frac{2\sqrt{x+1}-3}{3\sqrt{x+1}-2}\,dx$

(6) $\int \sqrt{2+\sqrt{x}}\,dx$

(7) $\int \frac{dx}{x(1-\sqrt[4]{x})}$

(8) $\int \frac{dx}{\sqrt{x}+\sqrt[3]{x}}$

(9) $\int \frac{x\,dx}{1+\sqrt{x}+x}$

(10) $\int \frac{dx}{\sqrt[3]{x}+\sqrt[4]{x}}$

(11) $\int (x^3-x)\sqrt{16-x^2}\,dx$

(12) $\int x^3\sqrt{x^2-9}\,dx$

(13) $\int \frac{x^5+2x^3}{\sqrt{x^2+4}}\,dx$

(14) $\int \frac{\sqrt{4+x^2}}{x}\,dx$

(15) $\int \frac{\sqrt{x^2+9}}{x^3}\,dx$

Exercises of Method [9]:

(1) $\int \frac{dx}{5+4\cos x}$

(2) $\int \frac{dx}{2-\sin x}$

(3) $\int \frac{\cos x\, dx}{5+4\cos x}$

(4) $\int \frac{dx}{2-\cos x}$

(5) $\int \frac{dx}{3+2\cos x+2\sin x}$

(6) $\int \frac{dx}{\tan x-\sin x}$

(7) $\int \frac{dx}{\cos x+\cot x}$

(8) $\int_0^{\pi} \frac{dx}{1+\sin x}$

(9) $\int_{\frac{\pi}{2}}^{\pi} \frac{dx}{1-\cos x}$

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Exercises: "Area under the curve"

1- Find the area bounded by the x-axis and the curve $y = x \sin x$ for: a) $0 \leq x \leq \pi$ Ans.: π unit^2

b) $\pi \leq x \leq 2\pi$ Ans.: 3π unit^2

2- Find the area between the curve $y = x(3 - x^2)^{1/2}$ and the x-axis. Ans.: $2\sqrt{3}$ unit^2

3- Find the area bounded by the curve $\sqrt{x} + \sqrt{y} = 1$ and the Coordinates axis. Ans.: $\frac{1}{6}$

4- Find the area in the 1st quadrant bounded by the curve $y = \frac{2}{x}$ and the lines $x = 2$, $y = 2$. Ans: $2(1 - \ln 2)$

5- Find the area under the curve $y = e^x \sin(e^x - 1)$, from $x = 0$ to $x = \ln(\pi + 1)$. Ans.: 2

6- Find the area bounded by $x = \sec y$, $x = 0$, $y = 0$ and $y = \frac{\pi}{3}$ Ans.: $\ln(2 + \sqrt{3})$

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Exercises: "Volumes"

1- Find the volume generated by revolving the region in the 1st quadrant bounded by $y = \tan^{-1}x$, $y=0$, $x=1$ about the y-axis

Ans.: $\pi\left(\frac{\pi}{2}-1\right)$

2- Find the volume swept out by revolving about the x-axis the region in the 1st quadrant bounded by $y = \frac{1}{\sqrt{1+x^2}}$ and $x=0$, $y=\frac{x}{\sqrt{2}}$.

Ans.: $\frac{\pi^2}{4} - \frac{\pi}{6}$

3- Find the volume generated by rotating about the x-axis the area in the 1st quadrant bounded by $y=\frac{1}{\sqrt{x}}$, the x-axis, $x=1$, $x=2$.

Ans.: $\pi \ln 2$

4- Find the volume of the solid generated by revolving the region bounded by $y=\frac{4}{x}$, $y=(x-3)^2$ about x-axis.

5- The region between the curve $y=e^{-x}$ and the x-axis from $x=1$ to $x=\ln 100$ is revolved about the x-axis. Find the volume of generated solid.

Ans.: $\frac{\pi}{2}(e^{-2}-0.01)$

6- Find the volume generated by rotating about the x-axis the area bounded by $y=e^x$, $y=0$, $x=0$, $x=2$.

Ans.: $\frac{\pi}{2}(e^4-1)$

7- Find the volume generated when the area bounded by $x=\sec y$, $x=0$, $y=0$, $y=\frac{\pi}{3}$ is rotated about the y-axis

Ans.: $\sqrt{3}\,\pi$

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Exercises: "Arc Length"

-Find the Arc length of the following curves:

1- $y = \frac{1}{6}x^3 + \frac{1}{2x}$, $1 \leq x \leq 3$

2- $y = \ln(\sec x)$, $0 \leq x \leq \frac{\pi}{3}$

3- $x = \ln(\sin y)$, $\frac{\pi}{4} \leq y \leq \frac{3\pi}{4}$

4- $x = \frac{1}{3}t^3$, $y = \frac{1}{2}t^2$, $1 \leq t \leq 2$

5- $x = e^{-t}\cos t$, $y = e^{-t}\sin t$, $0 \leq t \leq \frac{\pi}{2}$

6- $x = 2\cos t + 2t\sin t$, $y = 2\sin t - 2t\cos t$, $0 \leq t \leq \frac{\pi}{2}$

7- $y = \frac{1}{2}\cosh 2x$, $0 \leq x \leq 4$

8- $y = \frac{2}{3}x^{3/2} - \frac{1}{2}x^{1/2}$, $0 \leq x \leq 4$

9- $x = \frac{3}{5}y^{5/3} - \frac{3}{4}y^{1/3}$, $0 \leq y \leq 1$

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